Astronomy 722: Radiative Transfer and Gas Dynamics

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Homework 3 Due 5:10PM 3/13

While I may have consulted with other students in the class regarding this homework, the solutions presented here are my own work. I understand that to get full credit, I have to show all the steps necessary to arrive at the answer, and unless it is obvious, explain my reasoning using diagrams and/or complete sentences.

Name Signature:

1. (35%) As we have learned in class, when an external force accelerates a charge $q$ with acceleration $a$, it emits Larmor radiation.

   (a) (10%) If the motion of the charge is periodic, using the work-energy theorem and integration by parts, show that the process of emitting radiation should cause the charge to feel a second force, also called the *radiation reaction*:

   $$ F_{\text{rad}} = \frac{2}{3} \frac{q^2}{c^3} \dot{a} $$

   Where $\dot{a}$, the time derivative of the acceleration vector, is also known as the *jerk*. *The above result is valid for non-periodic accelerations as well*, but that more general derivation is beyond the scope of this course.

   (b) (15%) Calculate the radiation reaction force vector in polar coordinates for a classical electron at a distance $R$ from the proton (i.e., a classical hydrogen atom). To do this, remember that the time derivative of the unit vectors in polar coordinates are more complicated than they are in cartesian coordinates. Assume the motion is confined to a plane.

   (c) (10%) So overall, there will be two forces on the electron: the acceleration due to the proton, and the radiation reaction. The radiation reaction makes the electron eventually collide with the proton in most, but not all cases. Without explicitly solving for its motion, give an example of initial conditions where the electron does not collide with the proton.

2. (65%) At $t = 0$, an electron with zero total mechanical energy is falling along a straight line from a distance of 1 AU towards a million solar mass Schwarzschild black hole. While doing this, it emits light via Larmor radiation. This is a simple model of a radiatively inefficient accretion flow. Neglect relativistic effects.

   (a) (5%) Show that to very high accuracy, the radiation reaction force in Problem 1 has negligible impact on the electron’s motion, either with respect to direction, or with respect to the time taken to reach the even horizon of the black hole.

   (b) (5%) Find the initial velocity of the electron at $t = 0$. Hint: it’s not zero.

   (c) (20%) Solve for $r(t)$, the motion of the electron as a function of time. Calculate the time in seconds at which the electron goes through the event horizon of the black hole. *(In the GR treatment, the electron would disappear from our view as its light becomes infinitely redshifted a small distance before the horizon, but this solution is otherwise the correct relativistic result from the point of view of the electron!)*
The Fourier transform differentiation rule says that if $x(t)$ has a Fourier transform $\hat{x}(\omega)$, then $\dot{x}(t)$ has a Fourier transform equal to $i\omega\hat{x}(\omega)$, and $\ddot{x}(t)$ has a Fourier transform equal to $-\omega^2\hat{x}(\omega)$. However, this is not always true—explain the general conditions under which these rules fail and why.

Calculate the total emitted spectrum of the electron, $dW/d\omega$, from the moment it passes 1 AU to the moment it disappears from view. Argue based on part (c) that you cannot use equations 3.25-3.26 to do this calculation, but must directly take the Fourier transform of equation 3.24.

Plot the spectrum in a log-log plot of power per unit frequency vs. frequency. The spectrum should look like a power law—find its index by examining the plot (or plotting a power law next to it).