Physics 725: Special and General Relativity

Equal-area maps

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September 1, 2009

The line element on a sphere with longitude ϕ and latitude λ is

$$dS^2 = a^2(d\lambda^2 + \cos\lambda^2 d\phi^2)$$

A little patch of area on the sphere has area

$$dA_{\mathrm{sphere}} = a^2 \cos\lambda \, d\lambda \, d\phi$$

Let us explore how to make equal area maps of the sphere. As an example, let's make an equal-area polar map, with coordinates

$$r = ar(\lambda); \psi = \phi.$$

This map is centered on the north pole, so $r(\pi/2)$ must equal 0. The corresponding differentials are

$$dr = a\frac{dr}{d\lambda}d\lambda; d\psi = d\phi$$

What is the meaning of an equal area map? An equal area map is a transformation of the sphere into a plane such that the area on the sphere of any little patch is equal to the *Euclidean* area on the map. (Note that if we were merely to insert the new $d\lambda$ and $d\phi$ differentials into the first two equations, we would be getting the line and area elements for the *sphere*, not the plane). The *Euclidean* area element in polar coordinates is

$$dA_{\mathrm{map}} = a^2 r dr \, d\psi$$

To have an equal area map, we must have $dA_{\mathrm{sphere}} = CdA_{\mathrm{map}}$, where C is a constant. This gives us

$$\cos\lambda \, d\lambda \, d\phi = Cr(\lambda)\frac{dr(\lambda)}{d\lambda}d\lambda d\phi$$

which simplifies to

$$\cos\lambda = Cr(\lambda)\frac{dr(\lambda)}{d\lambda}$$

Integrating both sides we have

$$sin(\lambda) + r_0 = \frac{C}{2}r(\lambda)^2$$

and therefore

$$r(\lambda) = \sqrt{\frac{2}{C}(\sin\lambda + r_0)}$$

To satisfy $r(\pi/2) = 0$ we need to set our constant of integration $r_0 = -1$.

$$r(\lambda) = \sqrt{\frac{2}{C}(\sin\lambda - 1)}$$

Finally, it's clear that for r to be real C has to be negative, so we can write

$$r(\lambda) = \sqrt{\frac{2}{|C|}(1 - \sin\lambda)}$$