

Physics 725: Special and General Relativity

The L^2 Equations and the Christoffel Symbols

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To calculate orbits in General Relativity, we must find paths of extremal proper time,

$$\tau = \int \sqrt{-ds^2} = \int \left(-g_{\alpha\beta} dx^\alpha dx^\beta \right)^{\frac{1}{2}} = \int d\sigma \left(-g_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} \right)^{\frac{1}{2}} = \int L d\sigma,$$

where the variational calculus is done with respect to some parameter σ along the world line of the particle. The path of extremal proper time can be obtained by treating the integrand as the Lagrangian, and using the Lagrange equations

$$-\frac{d}{d\sigma} \left[\frac{\partial L}{\partial \left(\frac{dx^\alpha}{d\sigma} \right)} \right] + \frac{\partial L}{\partial x^\alpha} = 0$$

With the equality $d(L^2) = 2LdL$, these equations can be rewritten as

$$-\frac{d}{d\sigma} \left[\frac{1}{L} \frac{\partial L^2}{\partial \left(\frac{dx^\alpha}{d\sigma} \right)} \right] + \frac{1}{L} \frac{\partial L^2}{\partial x^\alpha} = 0 \quad (1)$$

And now using $L = \frac{d\tau}{d\sigma}$ we write this as

$$-\frac{d}{d\tau} \left[\frac{1}{L} \frac{\partial L^2}{\partial \left(\frac{dx^\alpha}{d\sigma} \right)} \right] + \frac{1}{L^2} \frac{\partial L^2}{\partial x^\alpha} = 0 \quad (2)$$

Now, taking a look at the actual definition of L , we can make use of the fact that the metric $g_{\alpha\beta}$ is *only* a function of the space variable x^α , and *not* a function of the velocity variables $dx^\alpha/d\sigma$. Specifically, since $d\tau = Ld\sigma$,

$$\frac{\partial L^2}{\partial \left(\frac{dx^\alpha}{d\sigma} \right)} = -2g_{\alpha\beta} \frac{dx^\beta}{d\sigma} = -2g_{\alpha\beta} \frac{dx^\beta}{d\tau} \frac{d\tau}{d\sigma} = -2Lg_{\alpha\beta} \frac{dx^\beta}{d\tau} = L \frac{\partial L^2}{\partial \left(\frac{dx^\alpha}{d\sigma} \right)} \Bigg|_{\sigma \rightarrow \tau} \quad (3)$$

If you don't understand where the "2" comes from, see the separate handout entitled "Where does the 2 come from?" For the second term, we have

$$\frac{\partial L^2}{\partial x^\alpha} = -\frac{\partial g_{\beta\gamma}}{\partial x^\alpha} \frac{dx^\beta}{d\sigma} \frac{dx^\gamma}{d\sigma} = -\frac{\partial g_{\beta\gamma}}{\partial x^\alpha} \frac{dx^\beta}{d\tau} \frac{d\tau}{d\sigma} \times \frac{dx^\gamma}{d\tau} \frac{d\tau}{d\sigma} = -L^2 \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} = L^2 \frac{\delta L^2}{\delta x^\alpha} \Bigg|_{\sigma \rightarrow \tau} \quad (4)$$

Why did we switch from $\alpha\beta$ to $\beta\gamma$ in the metric? Because we need to maintain x^α as our free variable, the variable whose orbit we are calculating. Now that we are differentiating with respect to x^α , the other indices need to change in order to not violate the Einstein summation convention (note that this wasn't the case with the velocities above, because the differentiation with respect to $dx^\alpha/d\sigma$ could be carried out without violating the summation convention).

Combining (2), (3), and (4) we obtain the **L^2 equation** for the motion of a particle in GR:

$$\boxed{-\frac{d}{d\tau} \left\{ \frac{\partial L^2}{\partial \left(\frac{dx^\alpha}{d\sigma} \right)} \Bigg|_{\sigma \rightarrow \tau} \right\} + \frac{\partial L^2}{\partial x^\alpha} \Bigg|_{\sigma \rightarrow \tau} = 0} \quad (5)$$

In my opinion, the above equation is the most straightforward way to derive equations of motion from a line element. The standard formulation takes this further, towards the calculation of the Christoffel symbols. Inserting equation (4) and (3) back into equation (5) gives us

$$-2 \frac{d}{d\tau} \left(g_{\alpha\beta} \frac{dx^\beta}{d\tau} \right) + \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} = 0 \quad (6)$$

which upon expanding the leftmost derivative becomes

$$-2 \frac{\partial g_{\alpha\beta}}{\partial \tau} \frac{dx^\beta}{d\tau} - 2g_{\alpha\beta} \frac{d^2 x^\beta}{d\tau^2} + \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} = 0$$

Now, it would be most convenient to transform derivatives of the metric with respect to τ into derivatives with respect to the coordinate variables x , because the metric is an explicit function of them. Therefore the first term in the above equation can be recast to yield:

$$-2 \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} \frac{dx^\gamma}{d\tau} \frac{dx^\beta}{d\tau} - 2g_{\alpha\beta} \frac{d^2 x^\beta}{d\tau^2} + \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} = 0$$

or, rearranging the terms on the other side,

$$g_{\alpha\beta} \frac{d^2 x^\beta}{d\tau^2} = \frac{1}{2} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} \left(\frac{\partial g_{\beta\gamma}}{\partial x^\alpha} - 2 \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} \right) \quad (7)$$

Now notice that *the term inside the parenthesis in the right hand side of the above isn't symmetric in β, γ* . We would like it to be symmetric, because the multiplied velocities $\frac{\delta x^\beta}{\delta \tau} \frac{\delta x^\gamma}{\delta \tau}$ are symmetric. We can easily recast it in a form that is symmetric by noticing first that:

$$\frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} = \frac{dx^\gamma}{d\tau} \frac{dx^\beta}{d\tau} \frac{\partial g_{\alpha\gamma}}{\partial x^\beta}$$

As a result,

$$2 \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} = \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{dx^\gamma}{d\tau} \frac{dx^\beta}{d\tau} \frac{\partial g_{\alpha\gamma}}{\partial x^\beta}$$

This gives:

$$g_{\alpha\beta} \frac{d^2 x^\beta}{d\tau^2} = \frac{1}{2} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} \left(\frac{\partial g_{\beta\gamma}}{\partial x^\alpha} - \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} \right) \quad (8)$$

Now, if we define the Christoffel symbols such that

$$\frac{d^2 x^\delta}{d\tau^2} = -\Gamma_{\beta\gamma}^\delta \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau}$$

or, multiplying both sides by $g_{\alpha\delta}$,

$$g_{\alpha\delta} \frac{d^2 x^\delta}{d\tau^2} = -g_{\alpha\delta} \Gamma_{\beta\gamma}^\delta \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau}$$

Comparing the above with equation (8), we have

$$\boxed{g_{\alpha\delta} \Gamma_{\beta\gamma}^\delta = \frac{1}{2} \left(\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} \right)} \quad (9)$$

An easy way to remember the above formula is to realize that the last term in the parenthesis (with the minus sign) goes with the derivative in α , the free index on the metric in the left hand side.

Another important point: in the above, going from equation (7) to equation (8) we split the $2 \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}$ term into $\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta}$. However, it is *not* in general true that $\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}$ should equal $\frac{\partial g_{\alpha\gamma}}{\partial x^\beta}$. They are different! Only the fact that the Christoffel symbols act as a sum over $\frac{\delta x^\beta}{\delta \tau} \frac{\delta x^\gamma}{\delta \tau}$, which is the same as the sum in equation (7), makes the step possible.