Physics 725: Special and General Relativity

Where Does the "2" Come from in the Lagrange Equations?

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In the Christoffel symbol handout, for the Lagrangian

$$L^2 = -g_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}$$

we said that the velocity derivative is

$$\frac{\partial L^2}{\partial\left(\frac{dx^\alpha}{d\sigma}\right)} = -2g_{\alpha\beta}\frac{dx^\beta}{d\sigma}$$

To see where the 2 comes from, let's write out all of L^2 by hand using ugly brute force:

$$\begin{aligned} -L^2 = g_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma} \quad = \quad & g_{00}\frac{dx^0}{d\sigma}\frac{dx^0}{d\sigma} + g_{01}\frac{dx^0}{d\sigma}\frac{dx^1}{d\sigma} + g_{02}\frac{dx^0}{d\sigma}\frac{dx^2}{d\sigma} + g_{03}\frac{dx^0}{d\sigma}\frac{dx^3}{d\sigma} + \\ & g_{10}\frac{dx^1}{d\sigma}\frac{dx^0}{d\sigma} + g_{11}\frac{dx^1}{d\sigma}\frac{dx^1}{d\sigma} + g_{12}\frac{dx^1}{d\sigma}\frac{dx^2}{d\sigma} + g_{13}\frac{dx^1}{d\sigma}\frac{dx^3}{d\sigma} + \\ & g_{20}\frac{dx^2}{d\sigma}\frac{dx^0}{d\sigma} + g_{21}\frac{dx^2}{d\sigma}\frac{dx^1}{d\sigma} + g_{22}\frac{dx^2}{d\sigma}\frac{dx^2}{d\sigma} + g_{23}\frac{dx^2}{d\sigma}\frac{dx^3}{d\sigma} + \\ & g_{30}\frac{dx^3}{d\sigma}\frac{dx^0}{d\sigma} + g_{31}\frac{dx^3}{d\sigma}\frac{dx^1}{d\sigma} + g_{32}\frac{dx^3}{d\sigma}\frac{dx^2}{d\sigma} + g_{33}\frac{dx^3}{d\sigma}\frac{dx^3}{d\sigma} \end{aligned}$$

Now try evaluating the derivative with respect to the 0th velocity; the corresponding terms in the sum are

$$\begin{aligned} -\frac{\partial L^2}{\partial\left(\frac{dx^0}{d\sigma}\right)} \quad = \quad & 2g_{00}\frac{dx^0}{d\sigma} + g_{01}\frac{dx^1}{d\sigma} + g_{02}\frac{dx^2}{d\sigma} + g_{03}\frac{dx^3}{d\sigma} + \\ & g_{10}\frac{dx^1}{d\sigma} + 0 + 0 + 0 \\ & g_{20}\frac{dx^2}{d\sigma} + 0 + 0 + 0 \\ & g_{30}\frac{dx^3}{d\sigma} + 0 + 0 + 0 \end{aligned}$$

But we know that the metric is a symmetric matrix, i.e.

$$g_{\alpha\beta} = g_{\beta\alpha}$$

As a result,

$$-\frac{\partial L^2}{\partial\left(\frac{dx^0}{d\sigma}\right)} \quad = \quad 2g_{00}\frac{dx^0}{d\sigma} + 2g_{01}\frac{dx^1}{d\sigma} + 2g_{02}\frac{dx^2}{d\sigma} + 2g_{03}\frac{dx^3}{d\sigma} = 2g_{0\beta}\frac{dx^\beta}{d\sigma}$$

And the same goes for all the other coordinates labeled 1, 2, and 3.

Let's emphasize the truly important point here: **the "2" is valid only for a symmetric metric.** Many of our results on the equations of motion would be invalid were the metric not symmetric.