Example using Bessel functions—Sp 2016

Circular wave guide

Let’s investigate the propagation of waves in a wave guide that has a circular cross-section of radius \(a\) filled with air. The \(z\)-axis runs along the cylinder axis. As usual we take

\[
\vec{E} \propto e^{ikz} e^{-i\omega t}
\]

to obtain the differential equation for the TM mode (waveguide notes eqn 20)

\[
(\nabla^2 + \gamma^2) E_z = 0
\]

Then (waveguide notes eqn 19 and Table on pg 8)

\[
\vec{E}_t = \frac{ik}{\gamma^2} \vec{v}_t E_z
\]

with (waveguide notes eqn 21, \(\varepsilon = \varepsilon_0, \mu = \mu_0\))

\[
\gamma^2 = \frac{\omega^2}{c^2} - k^2
\]

In polar coordinates the differential equation takes the form

\[
\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial E_z}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 E_z}{\partial \phi^2} + \gamma^2 E_z = 0
\]

We look for a solution of the form

\[
E_z = R(\rho) W(\phi)
\]

and separate:

\[
\frac{\rho}{R} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial R}{\partial \rho} \right) + \frac{1}{W} \frac{\partial^2 W}{\partial \phi^2} + \gamma^2 \rho^2 = 0
\]

The middle term is a function of \(\phi\) only, while the other two terms are a function of \(\rho\) only. Thus, setting the middle term equal to \(-m^2\), we get

\[
W = e^{im\phi}
\]

and the remaining equation for \(R\) is Bessel’s equation of order \(m\). (Lea eqn 8.69 with \(k \to \gamma\))

\[
\rho \frac{\partial}{\partial \rho} \left( \rho \frac{\partial R}{\partial \rho} \right) - m^2 R + \gamma^2 \rho^2 R = 0
\]

We need a solution that is finite at \(\rho = 0\), so we choose \(J\). Thus the solutions are

\[
E_{z,m} = J_m (\gamma \rho) e^{im\phi}
\]

The boundary condition is (waveguide notes eqn 9)

\[
E_z = 0 \text{ for } \rho = a
\]

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so $\gamma a$ must be one of the roots $x_{mn}$ of $J_m(x)$. Then

$$E_z = \sum_{m,n} A_{mn} J_m \left( \frac{x_{mn} \rho}{a} \right) e^{i m \phi} e^{i k z} e^{-i \omega t}$$  \hspace{1cm} (3)

We would need more information about how the guide is excited to find the coefficients $A_{mn}$.

We find the other field components from (equation 1)

$$\vec{E}_{i,mn} = \frac{i k}{\gamma^2} \vec{\nabla} E_{z,mn} = \frac{i k a^2}{x_{mn}^2} \left( \frac{\partial E_{z,mn}}{\partial \rho} \frac{\rho}{\rho} + \frac{1}{\rho} \frac{\partial E_{z,mn}}{\partial \phi} \frac{\phi}{\phi} \right)$$

$$= \frac{i k a^2}{x_{mn}^2} A_{mn} \left[ x_{mn} J'_m \left( \frac{x_{mn} \rho}{a} \right) e^{i m \phi} \frac{\rho}{\rho} + \frac{i m}{\rho} J_m \left( \frac{x_{mn} \rho}{a} \right) e^{i m \phi} \right] e^{i k z} e^{-i \omega t}$$

$$= \frac{k a}{x_{mn}} A_{mn} \left[ i J'_m \left( \frac{x_{mn} \rho}{a} \right) e^{i m \phi} \frac{\rho}{\rho} - \frac{m}{x_{mn}} \frac{a}{\rho} J_m \left( \frac{x_{mn} \rho}{a} \right) e^{i m \phi} \right] e^{i k z} e^{-i \omega t}$$

Taking the real part, we have the physical field:

$$\vec{E}_{mn} = - \frac{k a A_{mn}}{x_{mn}} \left[ J'_m \left( \frac{x_{mn} \rho}{a} \right) \sin(kz + m \phi - \omega t) \frac{\rho}{\rho} + \frac{m}{x_{mn}} \frac{a}{\rho} J_m \left( \frac{x_{mn} \rho}{a} \right) \cos(kz + m \phi - \omega t) \frac{\phi}{\phi} \right]$$

$$+ A_{mn} J_m \left( \frac{x_{mn} \rho}{a} \right) \cos(kz + m \phi - \omega t) \frac{\rho}{\rho}$$

The roots are: $x_{0n} = 2.4, 5.5, 8.6, \cdots$

$x_{1n} = 3.8, 7.0, \cdots$ etc (Jackson p114).

The lowest root is $x_{01} = 2.4$. Thus the cutoff frequency ($k = 0$) for the TM modes is (from eqn.2 with $k = 0$)

$$\frac{\omega_{\text{c,TM}}}{c} = \gamma_{\text{min}} = \frac{x_{\text{min}}}{a} = \frac{x_{01}}{a} = \frac{2.4}{a}$$

The lowest frequency ($m = 0$) mode has no $\phi$ dependence and $\vec{E}_1$ is purely radial. Since $J'_0 = -J_1$ and $J_1(0) = 0$, $\vec{E}_{1,0n} \to 0$ at the center of the guide, as it must. (Remember: field lines can’t cross.)

$$\vec{E}_{01} = A_{01} \left\{ J_0 \left( \frac{2.4 \rho}{a} \right) \hat{z} \cos(kz - \omega t) + \frac{k a}{2.4} J_1 \left( \frac{2.4 \rho}{a} \right) \hat{\rho} \sin(kz - \omega t) \right\}$$

with (eqn 2)

$$k = \sqrt{\frac{\omega^2}{c^2} - \frac{2.4 a^2}{a^2}}$$

The next highest cutoff is for $m = 1$, $n = 1$ with $\omega_{\text{c,11}} = 3.8 c/a$. The fields in the $m = 1$, $n = 1$ mode are

$$\vec{E}_{11} = A_{11} \left\{ \left[ \hat{z} - \frac{\hat{\rho} \cdot k a}{3.8} \right] J_0 \left( \frac{3.8 \rho}{a} \right) \cos(\phi + kz - \omega t) $$

$$- \frac{k a}{3.8} J_0 \left( \frac{3.8 \rho}{a} \right) - J_2 \left( \frac{3.8 \rho}{a} \right) \hat{\rho} \sin(\phi + kz - \omega t) \right\}$$

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where we used Lea eqn 8.90 for \( J'_0 \), and with
\[
k = \sqrt{\frac{\omega^2}{c^2} - \frac{3.8^2}{a^2}}
\]

At the guide center, \( \rho = 0 \), (to evaluate the \( \phi \) component, remember that \( J_1(x) = \frac{x}{2} + \cdots \) and take the limit of \( J_1(x)/x \) as \( x \to 0 \)).

\[
\vec{E}_{11}(0) = A_{11} \left\{ -\frac{ka}{3.8} \left( \frac{1}{2} \hat{\rho} \sin(\phi + kz - \omega t) + \frac{1}{2} \hat{\phi} \cos(\phi + kz - \omega t) \right) \right\}
\]
\[
= -\frac{ka}{7.6} A_{11} \left\{ \hat{\rho} [\sin \phi \cos (kz - \omega t) + \cos \phi \sin (kz - \omega t)] + \hat{\phi} [\cos \phi \cos (kz - \omega t) - \sin \phi \sin (kz - \omega t)] \right\}
\]
\[
= -\frac{ka}{7.6} A_{11} \left\{ \cos (kz - \omega t) \left( \hat{\rho} \sin \phi + \hat{\phi} \cos \phi \right) + \sin (kz - \omega t) \left( \hat{\rho} \cos \phi - \hat{\phi} \sin \phi \right) \right\}
\]
\[
= -\frac{ka}{7.6} A_{11} \left\{ \hat{y} \cos (kz - \omega t) + \hat{x} \sin (kz - \omega t) \right\}
\]

The field makes an angle \( kz - \omega t \) with the \( y \) axis and, at a fixed \( z \), rotates counter-clockwise in time.

If \( \omega > 3.8c/a \), \( \omega \) is also \( > 2.4c/a \), and so the \( m = 0, n = 1 \) mode will exist as well. However, we are still below the cutoff for the \( m = 0, n = 2 \) mode until \( \omega > 5.5c/a \).

Convince yourself that we have satisfied all the boundary conditions and that the fields make sense. I will leave it to you to calculate \( \vec{B} \).

The diagram shows \( \vec{E} \) in the \( m = 0, n = 1 \) mode. Note that the field lies along the axis at \( \rho = 0 \) and is perpendicular to the surface at \( \rho = a \).

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