A disk magnet of radius $a$ and height $h$, $h \ll a$, has uniform magnetization $\vec{M} = M\hat{z}$ throughout its interior. Find the magnetic fields $\vec{B}$ and $\vec{H}$.

As we have seen, (notes 1 equation 45) there is an effective magnetic "surface charge density" $\vec{M} \cdot \hat{n}$ at the surface of the magnet. In this case we have

$$\sigma_M = \vec{M} \cdot \hat{n} = M\hat{z} \cdot \hat{\rho} = M \cos \phi$$

at $\rho = a$. The magnetic scalar potential is then given by (eqn 46, notes 1)

$$\Phi_m (\rho, \phi, z) = \frac{1}{4\pi} \int_0^h \int_0^{2\pi} \frac{M \cos \phi'}{|\vec{x} - \vec{x}'|} a d\phi'$$

$$= \frac{1}{4\pi} \int_0^h \int_0^{2\pi} \frac{M \cos \phi'}{\sqrt{(z - z')^2 + \rho^2 + a^2 - 2\rho a \cos (\phi - \phi')}} a d\phi'$$

(1)

The integral is nasty in the general case. Later in the semester we shall develop tools that will allow us to find $\vec{H}$ everywhere. For now we will look at two special cases.
Outside the magnet at a great distance away, \( z \gg h \geq z' \), \( \rho \gg a \), we have

\[
\Phi_m = \frac{h}{4\pi} \int_0^{2\pi} \frac{M \cos \phi'}{\sqrt{z^2 + \rho^2 - 2\rho a \cos(\phi - \phi')}} \, d\phi'
\]

\[
= \frac{h}{4\pi R} \int_0^{2\pi} \frac{M \cos \phi'}{\sqrt{1 - 2 \frac{\rho a}{R^2} \cos(\phi - \phi')}} \, d\phi'
\]

\[
= \frac{M h}{4\pi R} \int_0^{2\pi} \cos \phi' \left\{ 1 + \frac{\rho a}{R^2} \cos(\phi - \phi') + \cdots \right\} \, d\phi'
\]

The first term integrates to zero. We expand the cosine in the second term, and make use of the orthogonality of the trig functions to get:

\[
\Phi_m = \frac{M h}{4\pi R} \int_0^{2\pi} \left[ \cos \phi' \frac{\rho a}{R^2} \left( \cos \phi \cos \phi' + \sin \phi \sin \phi' \right) + \cdots \right] \, d\phi'
\]

\[
= \frac{M h}{4\pi R^3} \pi a^2 \rho \cos \phi = \frac{M h \pi a^2}{4\pi} \frac{x}{\left( z^2 + \rho^2 \right)^{3/2}}
\]

With the magnetic moment \( \vec{m} \) equal to \( \vec{M}V = \pi a^2 hM \hat{x} \), the potential is

\[
\Phi_m = \frac{\vec{m} \cdot \vec{r}}{4\pi R^3}
\]

as expected for a dipole. (Compare with eqns 28 and 44 in Notes 1). In this region \( \vec{B} = \mu_0 \vec{H} \) because \( \vec{M} \) is zero outside the magnet.

Inside the magnet and near the axis, \( z - z' \ll a \), \( \rho \ll a \), we may expand eqn (1) in a different way. Letting \( R^2 = (z - z')^2 + \rho^2 \), and keeping terms up to the square of \( \rho/a \) and \( R/a \), we have

\[
\Phi_m = \frac{1}{4\pi} \int_0^{\rho} \int_0^{2\pi} M \cos \phi' \frac{a}{a} \left\{ 1 - \frac{R^2}{2a^2} \frac{2\rho a \cos(\phi - \phi')}{a^2} \right. \\
+ \frac{3}{8} \left[ \frac{2 \rho a \cos(\phi - \phi')}{a} \right]^2 + \cdots \left\} \, d\phi' \, dz'
\]

The first two terms integrate to zero, and the remainder are independent of \( z' \),
leaving

\[ \Phi_m \simeq \frac{hM}{4\pi} \int_0^{2\pi} \cos \phi' \left\{ \frac{\rho}{a} \left( \cos \phi \cos \phi' + \sin \phi \sin \phi' \right) \\
+ \frac{3}{2} \left( \frac{\rho}{a} \right)^2 \left( \cos \phi \cos \phi' + \sin \phi \sin \phi' \right)^2 \cdots \right\} \, d\phi' \]

\[ = \frac{hM}{4a} x + 0 + \text{correction terms} \]

where again we used the orthogonality of the trig functions. The linear potential gives a uniform field:

\[ \vec{H} = -\vec{\nabla} \Phi_m = -\frac{hM}{4a} \hat{x} \]

Note that \( \vec{H} \) is opposite \( \vec{M} \). The magnetic induction \( \vec{B} \) is

\[ \vec{B} = \mu_0 \left( \vec{H} + \vec{M} \right) = \mu_0 \vec{M} \left( 1 - \frac{h}{4a} \right) \]

and is in the same direction as \( \vec{M} \).

Because \( \vec{\nabla} \cdot \vec{B} = 0 \), the lines of \( \vec{B} \) form closed loops. On the other hand, \( \vec{H} \) diverges from the positive and negative sources on the two sides of the magnet. Thus \( \vec{H} \) is opposite \( \vec{B} \) in the interior.