1 Polarization

Here we want to investigate the vector nature of the E&M fields in a wave. As usual, Maxwell’s equations tell the whole tale. In a source-free region:

\[
\begin{align*}
\nabla \cdot \vec{E} &= 0 \\
\nabla \cdot \vec{B} &= 0 \\
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\nabla \times \vec{B} &= \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}
\end{align*}
\]

If we now assume that each field has the form \( \vec{E} = \vec{E}_0 e^{i(k \cdot \vec{r} - \omega t)} \), then we find:

\[
\begin{align*}
\vec{k} \cdot \vec{E} &= 0 \\
\vec{k} \cdot \vec{B} &= 0
\end{align*}
\]

i.e. both \( \vec{E} \) and \( \vec{B} \) are perpendicular to \( \vec{k} \), and:

\[
\vec{k} \times \vec{E} = \omega \vec{B}
\]

so \( \vec{B} \) is also perpendicular to \( \vec{E} \). Now if we put the \( z \)-axis along \( \vec{k} \), \( \vec{k} = k \hat{z} \), then we may express the wave amplitudes as:

\[
\vec{E}_0 = E_1 \hat{x} + E_2 \hat{y}
\]

and similarly for \( \vec{B}_0 \). Finally we may allow for phase shifts by letting \( E_j = E_{j0} e^{i\phi_j} \). Since \( \vec{E} \) and \( \vec{B} \) are perpendicular, and have related magnitudes (by eqn (1)), then

\[
B_2 = \frac{k}{\omega} E_1 \quad \text{and} \quad B_1 = -\frac{k}{\omega} E_2
\]

so we can focus attention on the components of \( \vec{E}_0 \).

1.1 Linear polarization

Remember that the real physical field is the real part of the complex number. If \( \phi_1 = \phi_2 \), then both components of \( \vec{E} \) vary in phase, and \( E_1/E_2 = E_{10}/E_{20} \) is constant in time and space. Thus the electric field vector has a constant direction as its magnitude varies. The wave is linearly polarized.
1.2 Elliptical polarization

If $\phi_1$ and $\phi_2$ differ by $\frac{\pi}{2}$, then at $z = 0$ we find:

$$\mathbf{E} = \text{Re}(E_{10}e^{i(\phi_1 - \omega t)}\hat{x} + E_{20}e^{i(\phi_2 - \omega t)}\hat{y})$$

(2)

As $t$ increases from $\phi_1/\omega$, $E_x$ decreases and $E_y$ increases: The electric field vector rotates counter-clockwise, and the tip of the vector traces out an ellipse as the vector rotates. This is an elliptically polarized wave. When the amplitudes $E_{10}$ and $E_{20}$ are equal, the curve is a circle and we have a right-hand circularly polarized wave. Stick the thumb of your right hand in the direction of propagation (the $z$-direction in this case) and your fingers curl in the direction that $\mathbf{E}$ rotates. If instead we take $\phi_2 = \phi_1 + \pi/2$, then we get:

$$\mathbf{E} = E_{10} \cos(\omega t - \phi_1) \hat{x} - E_{20} \sin(\omega t - \phi_1) \hat{y}$$

and the $y-$component becomes increasingly negative: i.e. the vector rotates clockwise. When the amplitudes $E_{10}$ and $E_{20}$ are equal, the curve is a circle and we have left-hand circular polarization.

1.3 Circular polarization

In the case of circular polarization, with $E_{10} = E_{20} = E_0/\sqrt{2}$, we have

$$\mathbf{E} = E_0 \left( \frac{\hat{x} \pm i\hat{y}}{\sqrt{2}} \right) e^{i(\phi_1 - \omega t)}$$

(3)

so it is convenient to use the complex polarization vectors

$$\hat{e}_{RL} = \frac{\hat{x} \pm i\hat{y}}{\sqrt{2}}$$

(4)

which have the properties

$$\hat{e}_R \cdot \hat{e}_L^* = 0 \quad \text{and} \quad \hat{e}_R \cdot \hat{e}_L^* = \hat{e}_L \cdot \hat{e}_L^* = 1$$

1.4 General case

Any wave may be decomposed into a sum of linearly polarized waves or a sum of circularly polarized waves. In the most general case we have elliptical polarization with the axes of the ellipse oriented at an angle $\theta$ to the $x$ and $y$ axes, where $\theta$ is unknown for the moment. Given $E_{10}$, $E_{20}$, $\phi_1$ and $\phi_2$, we’d like to find the shape and orientation of the ellipse. It’s most convenient to do the analysis using the circular polarization vectors (4). Let’s rewrite
\( \vec{E} \) (eqn 2) in terms of \( \hat{e}_R \) and \( \hat{e}_L \):

\[
\vec{E} = E_{10} e^{i(\phi_1 - \omega t)} \hat{x} + E_{20} e^{i(\phi_2 - \omega t)} \hat{y}
\]

\[
= E_{10} e^{i(\phi_1 - \omega t)} \left( \frac{\hat{e}_R + \hat{e}_L}{\sqrt{2}} \right) + E_{20} e^{i(\phi_2 - \omega t)} \left( \frac{\hat{e}_R - \hat{e}_L}{\sqrt{2}i} \right)
\]

\[
= \left[ (E_{10} e^{i\phi_1} - iE_{20} e^{i\phi_2}) \left( \frac{\hat{e}_R}{\sqrt{2}} \right) + (E_{10} e^{i\phi_1} + iE_{20} e^{i\phi_2}) \left( \frac{\hat{e}_L}{\sqrt{2}i} \right) \right] e^{-i\omega t}
\]

\[
= (E_R \hat{e}_R + E_L \hat{e}_L) e^{-i\omega t}
\]

where

\[
E_R = \frac{1}{\sqrt{2}} \left[ E_{10} \cos \phi_1 + E_{20} \sin \phi_2 + i \left( E_{10} \sin \phi_1 - E_{20} \cos \phi_2 \right) \right] = |E_R| e^{i\chi_R}
\]

The magnitude is given by

\[
|E_R| = \frac{1}{\sqrt{2}} \sqrt{(E_{10} \cos \phi_1 + E_{20} \sin \phi_2)^2 + (E_{10} \sin \phi_1 - E_{20} \cos \phi_2)^2}
\]

\[
= \frac{1}{\sqrt{2}} \sqrt{E_{10}^2 + E_{20}^2 + 2E_{10}E_{20} \sin (\phi_2 - \phi_1)}
\]

and the phase by

\[
\tan \chi_R = \frac{E_{10} \sin \phi_1 - E_{20} \cos \phi_2}{E_{10} \cos \phi_1 + E_{20} \sin \phi_2}
\]

For the left-hand component, we have

\[
E_L = \frac{1}{\sqrt{2}} \left[ E_{10} \cos \phi_1 - E_{20} \sin \phi_2 + i \left( E_{10} \sin \phi_1 + E_{20} \cos \phi_2 \right) \right] = |E_L| e^{i\chi_L}
\]

with

\[
|E_L| = \frac{1}{\sqrt{2}} \sqrt{E_{10}^2 + E_{20}^2 - 2E_{10}E_{20} \sin (\phi_2 - \phi_1)}
\]

and

\[
\tan \chi_L = \frac{E_{10} \sin \phi_1 + E_{20} \cos \phi_2}{E_{10} \cos \phi_1 - E_{20} \sin \phi_2}
\]

If \( \phi_2 - \phi_1 = \pi/2 \), then \( |E_R| = E_{10} + E_{20} \) and \( |E_L| = |E_{10} - E_{20}| \). Additionally, if \( E_{10} = E_{20} \), \( |E_L| = 0 \) as expected for right circular polarization.

We may factor (5) to get:

\[
\vec{E} = |E_R| \left[ \hat{e}_R + \hat{e}_L \frac{|E_L|}{|E_R|} \exp i \left( \chi_L - \chi_R \right) \right] \exp i(\chi_R - \omega t)
\]

\[
= |E_R| \left[ \hat{e}_R + \hat{e}_L \exp i \alpha \right] \exp i(\chi_R - \omega t)
\]

where the last line defines \( \varepsilon \) and \( \alpha \). In particular, using

\[
\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B},
\]
along with (7) and (9), we have
\[
\alpha = \chi_L - \chi_R = \tan^{-1} \frac{E_{10} \sin \phi_1 + E_{20} \cos \phi_2 - \tan^{-1} E_{10} \sin \phi_1 - E_{20} \cos \phi_2}{E_{10} \cos \phi_1 - E_{20} \sin \phi_2}
\]
\[
= \tan^{-1} \frac{E_{10} \sin \phi_1 + E_{20} \cos \phi_2}{1 + E_{10} \cos \phi_1 + E_{20} \cos \phi_2 - E_{10} \sin \phi_1 - E_{20} \sin \phi_2}
\]
\[
= \tan^{-1} \frac{2E_{10}E_{20} \cos (\phi_2 - \phi_1)}{E_{10}^2 - E_{20}^2}
\]  
Equation (11)

Expanding the circular polarization vectors in (10), we get
\[
\vec{E} = \frac{|E_R|}{\sqrt{2}} \left[ (1 + \varepsilon e^{i\alpha}) \hat{x} + i (1 - \varepsilon e^{i\alpha}) \hat{y} \right] e^{i(\chi_R - \omega t)}
\]  
Equation (12)

The physical field is the real part of (12):
\[
\vec{E}_{\text{phys}} = \frac{|E_R|}{\sqrt{2}} \left\{ [\cos (\chi_R - \omega t) + \varepsilon \cos (\alpha + \chi_R - \omega t)] \hat{x} - [\sin (\chi_R - \omega t) - \varepsilon \sin (\alpha + \chi_R - \omega t)] \hat{y} \right\}
\]  
Equation (13)

with magnitude
\[
|\vec{E}_{\text{phys}}| = \frac{|E_R|}{\sqrt{2}} \sqrt{[\cos (\chi_R - \omega t) + \varepsilon \cos (\alpha + \chi_R - \omega t)]^2 + [\sin (\chi_R - \omega t) - \varepsilon \sin (\alpha + \chi_R - \omega t)]^2}
\]
\[
= \frac{|E_R|}{\sqrt{2}} \sqrt{[\cos^2 (\chi_R - \omega t) + \varepsilon^2 \cos^2 (\alpha + \chi_R - \omega t)] + [\sin^2 (\chi_R - \omega t) + 2 \varepsilon \sin (\chi_R - \omega t) \cos (\alpha + \chi_R - \omega t) - 2 \varepsilon \sin (\alpha + \chi_R - \omega t) \sin (\chi_R - \omega t)]}
\]
\[
= \frac{|E_R|}{\sqrt{2}} \sqrt{1 + \varepsilon^2 + 2 \varepsilon \cos [\alpha + 2 (\chi_R - \omega t)]}
\]

Thus $|\vec{E}|$ is maximum when $\omega t = \alpha/2 + \chi_R$ and minimum when $\omega t = \alpha/2 + \pi/2 + \chi_R$. At the maximum, $\vec{E}$ makes angle $\theta$ with the $x$-axis, where, from (13),
\[
\tan \theta = \frac{E_y}{E_x} = \frac{\sin (\alpha/2) + \varepsilon \sin (\alpha/2)}{\cos (\alpha/2) + \varepsilon \cos (\alpha/2)} = \tan \frac{\alpha}{2}
\]  
Equation (14)

and thus we have elliptical polarization with major axis rotated through angle $\alpha/2$ from the $x$-axis. (See eqn (11). $\theta$ is zero if $|\phi_1 - \phi_2| = \pi/2$, as expected.) The ratio of major to minor axes is
\[
\frac{E_{\text{max}}}{E_{\text{min}}} = \frac{|E_R| + |E_L|}{|E_R| - |E_L|} = \frac{1}{\sqrt{1 - e^2}}
\]  
Equation (15)

where $e$ is the eccentricity.

1.5 Stokes Parameters

What do we actually observe? We can design our detectors to measure either linear or circular polarizations. From those observations, we would like to determine the polarization
characteristics of the observed radiation. That is, we will observe
\[ \hat{x} \cdot \vec{E} = E_x = a_1 e^{i \delta_1}, \quad \hat{y} \cdot \vec{E} = E_y = a_2 e^{i \delta_2} \]
or
\[ \hat{e}_R \cdot \vec{E} = E_R = a_{RE} e^{i \delta_R}, \quad \hat{e}_L \cdot \vec{E} = E_L = a_{LE} e^{i \delta_L} \]
(In our previous notation \( E_{10} = a_1 \) and \( \phi_1 = \delta_1 \), \( E_{20} = a_2 \) and \( \phi_2 = \delta_2 \).)

The Stokes parameters are defined as:
\[ s_0 = |E_x|^2 + |E_y|^2 = a_1^2 + a_2^2 = \text{total intensity} \]
\[ s_1 = |E_x|^2 - |E_y|^2 = a_1^2 - a_2^2 \]
\[ s_2 = 2 \text{Re}(E_x^* E_y) = 2a_1 a_2 \cos(\delta_2 - \delta_1) \]
\[ s_3 = 2 \text{Im}(E_x^* E_y) = 2a_1 a_2 \sin(\delta_2 - \delta_1) \]
with a similar set for the circular polarizations. The four parameters are not independent:
\[ s_2^2 = s_1^2 + s_2^2 + s_3^2 \]
They describe physical parameters as follows:
\[ \frac{s_1}{s_0} = \text{percent linear polarization} \]

From equation (11) we get
\[ \frac{s_2}{s_1} = \tan(2 \times \text{angle of polarization ellipse}) \]

The eccentricity of the polarization ellipse is given by (equation 15)
\[ e^2 = 1 - \left( \frac{|E_R| - |E_L|}{|E_R| + |E_L|} \right)^2 \frac{4a_{RE} a_{LE}}{(a_R + a_L)^2} \]
\[ = 4 \frac{1}{\sqrt{2}} \left[ a_1^2 + a_2^2 + 2a_1 a_2 \sin(\phi_2 - \phi_1) \right] \frac{1}{\sqrt{2}} \left[ a_1^2 + a_2^2 - 2a_1 a_2 \sin(\phi_2 - \phi_1) \right] \]
\[ = \frac{4 \sqrt{s_0^2 - s_3^2}}{[s_0 + s_3 + \sqrt{s_0 - s_3}]^2} \]
\[ = \frac{2 \sqrt{s_1^2 + s_2^2}}{s_0 + \sqrt{s_1^2 + s_2^2}} \]

Check: If \( \delta_1 = \delta_2 \) then \( s_2 = 2a_1 a_2 \) and
\[ \sqrt{s_1^2 + s_2^2} = \sqrt{(a_1^2 - a_2^2)^2 + 4a_1^2 a_2^2} = s_0 \]
So we get \( e = 1 \), as expected. The polarization is linear with no circularly polarized component.

See also Pacholczyk, Radio Astrophysics, Appendix I