Wave propagation
S.M. Lea
2016

1 A wave signal

An electromagnetic wave signal may be represented by a function of space and time $u(\vec{x}, t)$ where $u$ might be one component of the electric field, for example. Then we can write this function in terms of its Fourier transform:

$$u(\vec{x}, t) = \frac{1}{(2\pi)^2} \int_{\text{all } \omega \text{ and } \vec{k} \text{ space}} A(\vec{k}, \omega) \exp \left( i \vec{k} \cdot \vec{x} - i \omega t \right) d\omega d\vec{k}$$

The dispersion relation for the wave gives a relation (such as eqn 11 in the plasma wave notes) between $\omega$ and $\vec{k}$ which allows us to write the integral in terms of $\vec{k}$ alone:

$$u(\vec{x}, t) = \frac{1}{(2\pi)^2} \int_{\text{all } \vec{k} \text{ space}} A(\vec{k}) \exp \left[ i \vec{k} \cdot \vec{x} - i \omega \left( \vec{k} \right) t \right] d\vec{k}$$

Now let’s simplify by choosing the $x-$axis along the direction of propagation and letting the whole problem be one-dimensional, so that $u = u(x, t)$ (We also lose two factors of $\sqrt{2\pi}$.)

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} A(k) \exp \left[ ikx - i \omega (k) t \right] dk$$  (1)

The Fourier amplitude $A(k)$ is found from the value of $u$ at some $x$ and $t$, usually at $t = 0$. Often we can make use of the fact that $A(-k) = A(k)^*$ if $u$ is real, (see below) but this requires that $\omega$ also be real, which, in general, it may not be. So we have to take

$$u(x, t) = \text{Re} \left\{ \frac{1}{2\pi} \int_{-\infty}^{+\infty} A(k) \exp \left[ ikx - i \omega (k) t \right] dk \right\}$$

$$= \frac{1}{4\pi} \int_{-\infty}^{+\infty} \left\{ A(k) \exp \left[ ikx - i \omega (k) t \right] + A(k)^* \exp \left[ i \omega (k)^* t - ikx \right] \right\} dk$$  (2)

Now we let $\kappa = -k$ in the second term.

$$u(x, t) = \frac{1}{4\pi} \left\{ \int_{-\infty}^{+\infty} A(k) \exp \left[ ikx - i \omega (k) t \right] dk - \int_{-\infty}^{+\infty} A(-\kappa)^* \exp \left[ ikx + i \omega (-\kappa)^* t \right] d\kappa \right\}$$

$$= \frac{1}{4\pi} \int_{-\infty}^{+\infty} \left\{ A(k) \exp \left[ ikx - i \omega (k) t \right] + A(-k)^* \exp \left[ ikx + i \omega (-k)^* t \right] \right\} dk$$

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1 Effectively, $A(\vec{k}, \omega) = A(\vec{k}) \delta \left[ \omega - \omega \left( \vec{k} \right) \right]$, as in Lea Ch 7.
Then, evaluating at $\tau = 0$, we get
\begin{equation}
    u(\vec{x}, 0) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[ A(\vec{k}) + A(-\vec{k})^* \right] \exp(i \vec{k} \cdot \vec{x}) \, dk
\end{equation}

Inverting the transform:
\begin{equation}
    \frac{1}{2\sqrt{2\pi}} [A(k) + A(-k)^*] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u(x,0) \exp(-i k x) \, dx
\end{equation}

If $A(-k) = A(k)^*$, then
\begin{equation}
    A(k) = \int_{-\infty}^{+\infty} u(x,0) \exp(-i k x) \, dx
\end{equation}

But we have an additional initial condition. Taking the time derivative of (2), we have
\begin{align*}
    \frac{\partial}{\partial \tau} u(x,t) &= \frac{1}{4\pi} \int_{-\infty}^{+\infty} \left\{ -i \omega A(k) \exp[i k x - i \omega (k) t] + A(k)^* i \omega \exp(-i k x - i \omega t) \right\} \, dk \\
    \frac{\partial}{\partial \tau} u(x,0) &= \frac{1}{4\pi} \int_{-\infty}^{+\infty} -i \omega \left[ A(k) - A(-k)^* \right] \exp(i k x) \, dk
\end{align*}

Inverting, we have:
\begin{equation}
    -i \omega \frac{A(k) - A(-k)^*}{2} = \int_{-\infty}^{+\infty} \frac{\partial}{\partial \tau} u(x,0) \exp(-i k x) \, dx
\end{equation}

Combining the two relations (3) and (5) to eliminate $A(-k)^*$, we have
\begin{equation}
    A(k) = \int_{-\infty}^{+\infty} \left( u(x,0) - \frac{1}{i \omega} \frac{\partial}{\partial \tau} u(x,0) \right) \exp(-i k x) \, dx
\end{equation}

which is Jackson equation 7.91 modulo a factor of $\sqrt{2\pi}$. (This arises because I started with the transform over space and time, whereas J used the transform over space alone.) We regain equation (4) if $\partial u/\partial t|_{t=0} = 0$ at $t=0$.

\section{Gaussian Pulse}

To simplify the algebra in this section, let’s assume $\partial/\partial \tau u(x,0) = 0$. Generally the wave is sent at a “carrier frequency” $\omega_0$ with a corresponding $k_0$ so that $\omega_0 = \omega(k_0)$, and the Fourier amplitude $A(k)$ peaks at $k_0$. For example, a Gaussian pulse with a carrier frequency $\omega_0$ is written:
\begin{equation}
    u(x,0) = u_0 \cos(k_0 x) \exp\left(-\frac{x^2}{a^2}\right)
\end{equation}
Its transform is:

\[ A(k) = \int_{-\infty}^{+\infty} u_0 \cos(k_0x) \exp\left(-\frac{x^2}{a^2}\right) \exp(-ikx) \, dx \]

\[ = \int_{-\infty}^{+\infty} \frac{u_0}{2} (e^{ik_0x} + e^{-ik_0x}) \exp\left(-\frac{x^2}{a^2}\right) \exp(-ikx) \, dx \]

\[ = \frac{u_0}{2} \int_{-\infty}^{+\infty} \left[e^{-i(k-k_0)x} + e^{-i(k_0+k)x}\right] \exp\left(-\frac{x^2}{a^2}\right) \, dx \quad (8) \]

So we have two integrals of identical form. To do each, we complete the square:

\[ -\frac{x^2}{a^2} - i(k-k_0)x = -\frac{1}{a^2} \left[x^2 + i(k-k_0)ax + \left(i\frac{(k-k_0)a^2}{2}\right)^2 - \left(i\frac{(k-k_0)a^2}{2}\right)^2\right] \]

\[ = -\frac{1}{a^2} \left[i\frac{a^2(k-k_0)}{2} + x\right]^2 + \frac{1}{4}a^4(k-k_0)^2 \]

\[ = -\frac{1}{4}a^2(k-k_0)^2 - \frac{1}{a^2} \left[i\frac{a^2(k-k_0)}{2} + x\right]^2 \]

Thus the first term in the integral (8) is:

\[ \frac{u_0}{2} \exp\left(-\frac{a^2}{4}(k-k_0)^2\right) \int_{-\infty}^{+\infty} \exp\left[-\frac{1}{a^2}\left(i\frac{a^2(k-k_0)}{2} + x\right)^2\right] \, dx \]

\[ = \frac{u_0}{2} \exp\left(-\frac{a^2}{4}(k-k_0)^2\right) a \int_{-\infty+i\gamma}^{+\infty+i\gamma} \exp(-v^2) \, dv \]

\[ = \frac{u_0}{2} \exp\left(-\frac{a^2}{4}(k-k_0)^2\right) a\sqrt{\pi} \]

\[ = \frac{\sqrt{\pi}a u_0}{2} \exp\left(-\frac{a^2}{4}(k-k_0)^2\right) \]

where we set \( v = \left(i\frac{a^2(k-k_0)}{2} + x\right) \frac{1}{i} = i\gamma + x/a \), and used the fact that the integral of the Gaussian is independent of the path between \( \pm \infty \). (See Lea Ch 7 pg 328.) To obtain the second term we replace \( k_0 \) with \(-k_0\). Thus the result for \( A(k) \) is two Gaussians, centered at \( k = \pm k_0 \), and each of width \( 2/a \).

\[ A(k) = A_1(k) + A_2(k) = \sqrt{\pi}a \frac{u_0}{2} \left[ \exp\left(-\frac{a^2}{4}(k-k_0)^2\right) + \exp\left(-\frac{a^2}{4}(k+k_0)^2\right)\right] \quad (9) \]

### 3 Group velocity

For a Gaussian pulse, the integral in (1) has two terms. To evaluate the first term, we write \( \omega(k) \) in a Taylor series centered at \( k_0 \).

\[ \omega(k) = \omega_0 + (k-k_0) \frac{d\omega}{dk}\bigg|_{k_0} + \frac{1}{2}(k-k_0)^2 \frac{d^2\omega}{dk^2}\bigg|_{k_0} + \ldots \]
Substituting \( \omega (k) \) into the expression for \( u(x,t) \) (1),
\[
  u_1 = \int_{-\infty}^{+\infty} \frac{A_1(k)}{2\pi} \exp \left( i k x - it \left[ \omega_0 + (k - k_0) \frac{d\omega}{dk}_{k_0} + \left( \frac{k - k_0}{2} \right)^2 \frac{d^2\omega}{dk^2}_{k_0} + \ldots \right] \right) \, dk
\]
and dropping the term in \( \omega'' \), we have:
\[
  u_1(x,t) \approx \frac{1}{2\pi} \exp \left[ -i \left( \omega_0 - k_0 \frac{d\omega}{dk}_{k_0} \right) t \right] \int_{-\infty}^{+\infty} A_1(k) \exp \left( i k x - ik \frac{d\omega}{dk}_{k_0} t \right) \, dk + \ldots
\]
\[
  u_1(x,t) = \frac{1}{2\pi} \exp \left[ -i \left( \omega_0 - k_0 \frac{d\omega}{dk}_{k_0} \right) t \right] \int_{-\infty}^{+\infty} A_1(k) \exp \left[ ik \left( x - \frac{d\omega}{dk}_{k_0} t \right) \right] \, dk
\]
where we used eqn (1) again in the last step, with \( x \to x - \frac{d\omega}{dk}_{k_0} t \) and \( t \to 0 \). Thus to first order in the expansion of \( \omega(k) \), \( u(x,t) \) equals a phase factor \( e^{-i\phi} \) times \( u(x-v_g t,0) \), that is, at time \( t \) the signal looks like the pulse at \( t = 0 \) translated at speed \( v_g \), where
\[
  v_g = \frac{d\omega}{dk}_{k_0}
\]
is the group speed. The phase \( \phi = (\omega_0 - v_g k_0) t \) indicates that the carrier wave shifts within the Gaussian envelope.

To evaluate the second integral we expand \( \omega(k) \) in a Taylor series centered at \( -k_0 \), and obtain the same result.

4 Pulse spreading

For some waves with \( \frac{d^2\omega}{dk^2} = 0 \), there is only a first order term in the Taylor series for \( \omega(k) \). But for other waves the higher order terms produce corrections to the first order result. Generally these terms lead to both spreading and distortion of the pulse shape. For example, consider the Whistler (Plasmawaves notes eqn 33):
\[
  k = \frac{\omega_p}{c} \sqrt{\frac{\omega}{\Omega}}
\]
or
\[
  \omega = \Omega \frac{k^2 c^2}{\omega_p}
\]
The group velocity for this wave is:
\[
  v_g = \frac{d\omega}{dk} = 2\Omega \frac{k c^2}{\omega_p}
\]
(13)
and the second derivative is
\[ v'_g = \frac{d^2 \omega}{dk^2} = 2\Omega^2 \omega^2 \]  
(14)

All further derivatives are zero. Thus from (10), the exact expression for \( u \) in this case is:

\[
u (x, t) = \exp \left[ -i \left( \omega_0 - k_0 \frac{d\omega}{dk} \right) t \right] \times \]

\[
\frac{1}{2\pi} \int_{-\infty}^{+\infty} A(k) \exp \left[ ik (x - \frac{d\omega}{dk} t) - i (k - k_0)^2 \frac{d^2 \omega}{dk^2} \right] \frac{d\omega}{dk} \]  
\]

\[
= \exp \left[ -i \left( \omega_0 - k_0 v_g + k_0^2 v'_g \right) t \right] \times \]

\[
\frac{1}{2\pi} \int_{-\infty}^{+\infty} A(k) \exp \left[ ik \left( x \pm k_0 v'_g t - v_g t \right) - \frac{i}{2} k^2 v'_g t \right] \]  
\]

where \( v_g \) and \( v'_g \) are given by equations (13) and (14) with \( k = k_0 \).

Now if we use the Gaussian pulse (9) as an example, we get:

\[
u (x, t) = \exp \left[ -i \left( \omega_0 - k_0 v_g + \frac{k_0^2 v'_g}{2} \right) t \right] \frac{1}{2\pi I} \]  
(15)

where

\[
I = \sqrt{\pi} \int_{-\infty}^{+\infty} \frac{a u_0 \exp \left[ -\frac{a^2}{4} (k - k_0)^2 \right]}{\frac{1}{2} + \exp \left[ -\frac{a^2}{4} (k + k_0)^2 \right]} \exp \left[ ik \left( x \pm k_0 v'_g t - v_g t \right) - \frac{i}{2} k^2 v'_g t \right] \]  
\]

Again there are two terms, differing only in the sign of \( k_0 \) in the Gaussian, and we evaluate these using the Taylor series about \( \pm k_0 \).

Looking at the first term:

\[
I_+ = \int_{-\infty}^{+\infty} \exp \left[ -\frac{a^2}{4} (k - 2k_0 + k_0^2) + ik (x + k_0 v'_g t - v_g t) - \frac{i}{2} k^2 v'_g t \right] \]  
\]

\[
= \exp \left( -\frac{a^2 k_0^2}{4} \right) \int_{-\infty}^{+\infty} \exp \left[ -k^2 \left( \frac{a^2}{4} + \frac{i}{2} v'_g t \right) + k \left( i (x + k_0 v'_g t - v_g t) + \frac{k_0 a^2}{2} \right) \right] \]  
\]

To simplify the notation, we let \( s = x - v_g t \) and \( v'_g t = \alpha a^2 \). Then the exponent in the integrand is:

\[
-k^2 \left( \frac{a^2}{4} + \frac{i}{2} \alpha a^2 \right) + k \left( \alpha a^2 \right) + k_0 a^2 \]  
\]

\[
= -\frac{a^2}{4} \left( 1 + 2i\alpha \right) k^2 + -\frac{k_0 a^2}{2} \left( 1 + 2i\alpha \right) k \]  
\]

5
Completing the square, the term in square brackets is
\[
k^2 + 2k \left[ \frac{2is}{a^2(1 + 2i\alpha)} + k_0 \right] + \left[ \frac{2is}{a^2(1 + 2i\alpha)} + k_0 \right]^2 - \left[ \frac{2is}{a^2(1 + 2i\alpha)} + k_0 \right]^2
\]
\[
= \left[ k + \frac{2is}{a^2(1 + 2i\alpha)} + k_0 \right]^2 - \left[ \frac{2is}{a^2(1 + 2i\alpha)} + k_0 \right]^2
\]
We can do the integral by making the change of variable:
\[
\xi = \frac{a}{2} \sqrt{1 + 2i\alpha} \left[ k + \frac{2is}{a^2(1 + 2i\alpha)} + k_0 \right]
\]
Then:
\[
I_+ = \exp \left( -\frac{a^2k_0^2}{4} \right) \exp \left\{ \frac{a^2}{4} (1 + 2i\alpha) \left[ \frac{2is}{a^2(1 + 2i\alpha)} + k_0 \right]^2 \right\} \frac{2}{a\sqrt{1 + 2i\alpha}} \int_{-\infty+i\gamma}^{+\infty+i\gamma} e^{-\xi^2} d\xi
\]
where the path of integration is again moved off the real axis, but the result is still $\sqrt{\pi}$. Thus:
\[
I_+ = \frac{2\sqrt{\pi}}{a\sqrt{1 + 2i\alpha}} \exp \left\{ -\frac{a^2k_0^2}{4} + \frac{s^2}{a^2(1 + 2i\alpha)} + isk_0 + \frac{1}{2}ak_0^2\alpha \right\}
\]
And the exponent in $u$ (15) is:
\[
i \left( -\omega_0t + k_0x - \frac{k_0^2a^2\alpha}{2} + k_0s + \frac{1}{2}ak_0^2\alpha \right) - \frac{s^2}{a^2(1 + 4\alpha^2)}
\]
\[
= i \left( -\omega_0t + k_0x \right) - \frac{s^2}{a^2(1 + 4\alpha^2)} + \frac{2i\alpha s^2}{a^2(1 + 4\alpha^2)}
\]
Thus
\[
u (x, t) = \sqrt{\pi} \frac{a\omega_0}{2} \frac{1}{2\pi} \exp \left[ i \left( k_0x - \omega_0t \right) \right] \exp \left[ -\frac{s^2}{a^2(1 + 4\alpha^2)} \right] \times
\]
\[
\exp \left[ \frac{2s^2}{a^2(1 + 4\alpha^2)} \right] \frac{2\sqrt{\pi}}{a\sqrt{1 + 2i\alpha}} + (k_0 \to -k_0)
\]
\[
= u_0 \exp \left[ i \left( k_0x - \omega_0t \right) \right] \frac{2}{(1 + 4\alpha^2)^{1/4}} \exp \left[ -\frac{(x - vt)^2}{a^2(1 + 4\alpha^2)} \right] e^{-i\phi} + (k_0 \to -k_0)
\]
We may interpret this expression as follows:
- Initial wave form at frequency $\omega_0$. This is the carrier.
- Envelope peaks at $x = v_\gamma t$ due to its travelling at the group speed.
- Envelope spread by factor $\sqrt{1 + 4\alpha^2}$ where $\alpha = v_\gamma^2t/a^2$
- Amplitude decreased by similar factor
• Overall phase factor

\[
\phi = -\alpha \frac{2s^2}{a^2 (1 + 4\alpha^2)} + \frac{\tan^{-1}(2\alpha)}{2}
\]

approximately proportional to \( \alpha \propto t \).

At large times the pulse width grows roughly linearly with time. \( a(t) \approx 2a(0) \alpha = 2v_g t/a(0) \). Notice that (18) gives the correct result (7) at \( t = 0 \).

5 Arrival of a signal

The previous analysis shows the shape of the signal over all space as a function of time. But this is not usually what we observe. Let us now consider a signal generated at \( x = 0 \) over a period of time. We want to see what kind of a signal arrives at a distant point \( x = X \) as a function of time. So instead of using the dispersion relation in the form \( \omega(k) \) we instead think of it as \( k(\omega) \). Then:

\[
u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} A(\omega) \exp\left[ik(\omega)x - i\omega t\right] d\omega
\]

where in general \( k(\omega) \) is complex.

As we have seen, \( A(\omega) \) is usually a smooth function, somewhat peaked around the carrier frequency \( \omega_0 \). The exponential oscillates, so it tends to make the value of the integral small. However, when the exponent stays almost constant over a range of \( \omega \), we will have a substantial contribution to the integral. This happens when the phase is stationary. (See Lea Optional Topic D section 2)

\[
\frac{d}{d\omega} [k(\omega) x - \omega t] = 0
\]

\[
x \frac{dk}{d\omega} - t = 0
\]

or

\[
x = \frac{d\omega}{dk} t = v_g t
\]

(19)

Thus the major contribution to the integral is from the frequency that, at time \( t \), has its group speed equal to \( x/t \). Put another way, the signal at that frequency, travelling at the group speed, has just reached the observation point.

To find the received signal, we must evaluate the integral. Again we expand \( k(\omega) \) in a Taylor series, this time about the stationary frequency, as determined by equation (19).

\[
k(\omega) = k_s + (\omega - \omega_s) \frac{dk}{d\omega} \bigg|_{\omega_s} + \frac{1}{2} (\omega - \omega_s)^2 \frac{d^2k}{d\omega^2} \bigg|_{\omega_s} + \cdots
\]

where \( k_s = k(\omega_s) \), giving

\[
u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} A(\omega) \exp\left[i \left(k_s + (\omega - \omega_s) \frac{dk}{d\omega} \bigg|_{\omega_s} + (\omega - \omega_s)^2 \frac{d^2k}{d\omega^2} \bigg|_{\omega_s} + \cdots \right)x - i\omega t\right] d\omega
\]

Now use the stationary phase condition (19):
\[ u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} A(\omega) \exp \left\{ i \left[ k_x - \frac{\omega_s}{v_g} + \frac{1}{2} (\omega - \omega_s)^2 \frac{d^2k}{d\omega^2} \bigg|_{\omega_s} + \ldots \right] v_g t \right\} d\omega \]

Since the exponential guarantees that only frequencies near \( \omega_s \) contribute, we may pull out the slowly varying amplitude, as well as the terms in the exponential that are independent of \( \omega \):

\[ u(x,t) \simeq \frac{A(\omega_s)}{\sqrt{2\pi}} \exp \left[ i (k_x v_g - \omega_s) t \right] \int_{-\infty}^{+\infty} \exp \left[ i \frac{1}{2} (\omega - \omega_s)^2 \frac{d^2k}{d\omega^2} \bigg|_{\omega_s} v_g t \right] d\omega \]

To do the integral, change variables to:

\[ \xi = (\omega - \omega_s) \sqrt{-i} \frac{d^2k}{d\omega^2} \bigg|_{\omega_s} v_g t = e^{-i \pi/4} (\omega - \omega_s) \sqrt{\frac{d^2k}{d\omega^2} \bigg|_{\omega_s}} v_g t / 2 \]

Then

\[ I = \int_{-\infty}^{+\infty} \exp \left[ i \frac{\omega}{2} - \omega_s \right] \frac{d^2k}{d\omega^2} \bigg|_{\omega_s} v_g t \] \[ d\omega = \frac{e^{i \pi/4}}{\sqrt{\frac{d^2k}{d\omega^2} \bigg|_{\omega_s} v_g t}} \int_{-\infty}^{+\infty} e^{-\xi^2} d\xi \]

and so, incorporating the result (19), we have

\[ u(x,t) = \frac{A(\omega_s)}{\sqrt{\frac{d^2k}{d\omega^2} \bigg|_{\omega_s}}} \exp \left[ i (k_x v_g - \omega_s) t + i \frac{\pi}{4} \right] \] (20)

Again we see the carrier wave, modulated by an envelope that changes in time, and with an additional phase change. (Remember that \( \omega_s \) is a function of time.)

To see how this works out, again look at the whistler. The stationary phase condition is (equations 13 and 19)

\[ x = 2\Omega \frac{k c^2}{\omega_p} t = \frac{\omega_p}{c} \sqrt{\frac{\omega}{\Omega}} \frac{2\Omega c^2 t}{\omega_p} \]

\[ = 2 \frac{\sqrt{\omega \Omega}}{\omega_p} ct \]

and thus

\[ \omega_s = \frac{\omega_p^2}{\Omega} \frac{x^2}{4c^2 t^2} \]

which decreases from a large value toward zero as \( t \) increases. For this wave the signal (20) is given by:

\[ u(x,t) = \sqrt{\frac{2\Omega c^2}{\omega_p}} A(\frac{\omega_p}{\Omega} \frac{x^2}{4c^2 t^2}) \exp \left[ i (k_x v_g - \omega_s) t + i \frac{\pi}{4} \right] \]

\[ = \frac{c}{\omega_p} \frac{\sqrt{2\Omega}}{x} A(\frac{\omega_p^2}{\Omega} \frac{x^2}{4c^2 t^2}) \exp \left[ i (k_x v_g - \omega_s) t + i \frac{\pi}{4} \right] \]
If $A$ is a Gaussian as in (9), then

$$u(x,t) = \frac{c}{\omega_p} \sqrt{\frac{2\Omega}{x}} A_0 \exp \left[ -\eta \left( \frac{\omega_p^2}{\Omega} \frac{x^2}{4c^2 t^2} - \omega_0 \right) \right] \exp \left[ i (k_x v_g - \omega_s) t + i \frac{\pi}{4} \right]$$

where $\eta$ is a constant.

At fixed $x$, we get something like this:

![Graph of signal growth](image)

The signal grows from zero to an asymptotic value, until the analysis breaks down at low frequency (large $\tau$). (Remember: we neglected ion motion in getting the dispersion relation.)

Here’s another way to look at it. Consider the plasma waves (Plasmawave notes eqn 11):

$$c^2 k^2 = \omega^2 - \omega_p^2$$

For this wave

$$2c^2 k \frac{dk}{d\omega} = \frac{2\omega}{c \sqrt{\omega^2 - \omega_p^2}}$$

which looks like this:

![Graph of $k/d\omega$ vs $\omega$](image)
The stationary phase condition is
\[
\frac{ct}{x} = \frac{c}{d\omega} \frac{dk}{d\omega}
\]
There is no signal at a fixed \( x = x_0 \) prior to \( t = t_0 = x_0/c \). The signal begins at infinite frequency, and moves to lower frequency as time increases. The horizontal line in the graph represents \( ct/x = 2 \) and the intersection of the two lines is the observed frequency at this time.

For a more complicated dispersion relation, the signal can be more complicated. Consider the RHC wave propagated along \( \hat{B} \). (Plasmawave notes eqn 27 with the minus sign.) Then:

\[
\begin{align*}
\omega^2 k^2 &= \omega^2 - \frac{\omega^2}{\omega - \Omega} \\
2c^2 k \frac{dk}{d\omega} &= 2\omega - \frac{\omega_p^2}{\omega - \Omega} + \frac{\omega^2}{(\omega - \Omega)^2} \\
&= 2\omega - \frac{\omega_p^2}{\omega - \Omega} \left(1 - \frac{\omega}{\omega - \Omega}\right) \\
&= 2\omega + \frac{\omega_p^2}{(\omega - \Omega)^2}
\end{align*}
\]

and thus:

\[
\begin{align*}
dk &= \frac{1}{ck} \left(\omega + \frac{\omega_p^2}{2(\omega - \Omega)^2}\right) \\
&= \frac{1}{\sqrt{\omega^2 - \frac{\omega_p^2}{\omega - \Omega}}} \left[\omega + \frac{\omega_p^2}{2(\omega - \Omega)^2}\right] \\
&= \frac{1}{\sqrt{1 - \frac{\omega_p^2}{\omega - \Omega}}} \left[1 + \frac{\omega_p^2}{2(\omega - \Omega)^2}\right]
\end{align*}
\]

Now let \( \omega_p/\Omega = y \) and \( \omega/\Omega = z \) Then

\[
\begin{align*}
c \frac{dk}{d\omega} &= \frac{1}{\sqrt{1 - \frac{y^2}{z(z - 1)}}} \left(1 + \frac{1}{2} \frac{y^2}{z(z - 1)^2}\right)
\end{align*}
\]

With \( y = 2 \), the diagram looks like:
As $t$ increases we first obtain a signal at one frequency (dashed line) and later at 3 frequencies, two decreasing and one increasing (solid line).