Elasticity, Restoring Forces, & Oscillations / Pendulums

When a basketball bounces off the ground, there is an instant
when the ball is at rest, but energy is stored in the increased
air pressure in the ball, which pushes against the ground
to accelerate the ball back up. This is an example of a restoring
force.

The simplest restoring force varies linearly
with displacement from equilibrium.

This is expressed by Hooke's Law \( F = -kx \)
where \( x = 0 \) is the equilibrium point.

Springs are an example: their tension increases
linearly with how far they are stretched (unless they
are stretched too far). This is how a scale works.

Then \( k \) is the spring constant with units \( N/m \).

If the net force on a mass is a restoring force, then the mass
will oscillate in simple harmonic motion when displaced from equilibrium.

\[ \Sigma F = m \ddot{x} = -kx \]

This 2nd order diff. eq. has solution

\[ \ddot{x} = \frac{k}{m} x \]

\[ \dot{x} = A \cos(\omega t + \phi) \]

where amplitude \( A \) is the
magnitude of maximum displacement from equilibrium,
angular frequency \( \omega = 2\pi f = \frac{2\pi}{T} \) for period of oscillation \( T \), and
phase \( \phi = \phi_0 + \omega t \) with initial phase \( \phi_0 \) determined by \( x_0 = A \cos(\phi_0) \).

For a mass on a spring \( \omega = \sqrt{\frac{k}{m}} \). We can show that a mass hanging
vertically on a spring oscillates about its equilibrium point in shm,
although the spring is stretched a distance \( \frac{mg}{k} \) at equilibrium.

The \( + \) sign in \( \phi = \phi_0 + \omega t \) makes sure that for initial displacement down
\( x_0 = -A \Rightarrow \phi_0 = \pi \Rightarrow \dot{x}(t = \frac{T}{4}) = +v_{max} \): the velocity immediately after \( x_0 \) is up.

\( \dot{x} = \frac{d^2x}{dt^2} = -\omega^2 A \cos(\phi) \Rightarrow v_{max} = \omega A = 2\pi \frac{A}{T} \)

\( \ddot{x} = \frac{d^2x}{dt^2} = -\omega^2 A \cos(\phi) \Rightarrow \ddot{x} = -\omega^2 \frac{k}{m} x \)

where \( \omega = \sqrt{\frac{k}{m}} \) for a mass on a spring.

The Simple Pendulum - for small displacements from equilibrium
gravity acts as a restoring force on a mass hanging on a string.

\[ T = mg \cos \theta \]

\[ \ddot{\theta} = \Sigma \vec{F} = mg \sin \theta (-\dot{\theta}) \]

\[ = mg \frac{x}{L} (-\dot{\theta}) = \sqrt{\frac{g}{L}} \sin \theta (-\dot{\theta}) \]

\[ \ddot{\theta} = \frac{d^2 \theta}{dt^2} = -\frac{g}{L} \theta \Rightarrow \theta = L \theta_{max} \cos(\phi) \]

for a simple pendulum \( \omega = \sqrt{\frac{g}{L}} \)

and \( T = 2\pi \sqrt{\frac{L}{g}} \).
Energy in s.h.m. / Resonance & Damping

The total energy of a simple harmonic oscillator is conserved, alternating between kinetic energy and potential energy. For a mass on a spring $K = \frac{1}{2}mv^2$ and $U = \frac{1}{2}kx^2$ so the total energy is $E = K + U = \frac{1}{2}m(\omega^2x^2 + \frac{1}{2}k(x_0)^2) = \frac{1}{2}m\omega^2x^2 = \frac{1}{2}kA^2$ constant.

Damped Harmonic Motion

If an external force provides power to the mass, either by a continuous oscillating force or periodic impulse, then the total energy may increase. If the period of an oscillating force or impulse is the same as the period of the mass on the spring, then the mass on the spring will resonate in response to the external force. If unchecked, resonance leads to an infinite amplitude but in practice there is always a damping force that limits amplitude.

If the damping force is a linear drag force then

$\Sigma F = ma$

$-kx - bx = ma$

$\Rightarrow 0 = m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx$

$X = e^{-\frac{b}{2m}t}A\cos(\omega t + \phi_0)$

where $\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$

The damping force dissipates energy (converting it into heat), decreasing the amplitude of oscillation according to $Ae^{-\frac{b}{2m}t}$ and also slows the frequency of oscillation slightly.