Special Relativity

ON THE ELECTRODYNAMICS OF MOVING BODIES

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June 30, 1905
It is known that Maxwell’s electrodynamics—as usually understood at the present time—when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. For if the magnet is in motion and the conductor at rest, there arises in the neighbourhood of the magnet an electric field with a certain definite energy, producing a current at the places where parts of the conductor are situated. But if the magnet is stationary and the conductor in motion, no electric field arises in the neighbourhood of the magnet. In the conductor, however, we find an electromotive force, to which in itself there is no corresponding energy, but which gives rise—assuming equality of relative motion in the two cases discussed—to electric currents of the same path and intensity as those produced by the electric forces in the former case.
Examples of this sort, together with the unsuccessful attempts to discover any motion of the earth relatively to the “light medium,” suggest that the phenomena of electrodynamics as well as of mechanics possess no properties corresponding to the idea of absolute rest. They suggest rather that, as has already been shown to the first order of small quantities, the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good.\textsuperscript{1} We will raise this conjecture (the purport of which will hereafter be called the “Principle of Relativity”) to the status of a postulate, and also introduce another postulate, which is only apparently irreconcilable with the former, namely, that light is always propagated in empty space with a definite velocity $c$ which is independent of the state of motion of the emitting body. These two postulates suffice for the attainment of a simple and consistent theory of the electrodynamics of moving bodies based on Maxwell’s theory for stationary bodies. The introduction of a “luminiferous ether” will prove to be superfluous inasmuch as the view here to be developed will not require an “absolutely stationary space” provided with special properties, nor assign a velocity-vector to a point of the empty space in which electromagnetic processes take place.

The theory to be developed is based—like all electrodynamics—on the kinematics of the rigid body, since the assertions of any such theory have to do with the relationships between rigid bodies (systems of co-ordinates), clocks, and electromagnetic processes. Insufficient consideration of this circumstance lies at the root of the difficulties which the electrodynamics of moving bodies at present encounters.
I. KINEMATICAL PART

§ 1. Definition of Simultaneity

Let us take a system of co-ordinates in which the equations of Newtonian mechanics hold good. In order to render our presentation more precise and to distinguish this system of co-ordinates verbally from others which will be introduced hereafter, we call it the “stationary system.”

If a material point is at rest relatively to this system of co-ordinates, its position can be defined relatively thereto by the employment of rigid standards of measurement and the methods of Euclidean geometry, and can be expressed in Cartesian co-ordinates.

If we wish to describe the motion of a material point, we give the values of its co-ordinates as functions of the time. Now we must bear carefully in mind that a mathematical description of this kind has no physical meaning unless we are quite clear as to what we understand by “time.” We have to take into account that all our judgments in which time plays a part are always judgments of simultaneous events. If, for instance, I say, “That train arrives here at 7 o’clock,” I mean something like this: “The pointing of the small hand of my watch to 7 and the arrival of the train are simultaneous events.”
It might appear possible to overcome all the difficulties attending the definition of “time” by substituting “the position of the small hand of my watch” for “time.” And in fact such a definition is satisfactory when we are concerned with defining a time exclusively for the place where the watch is located; but it is no longer satisfactory when we have to connect in time series of events occurring at different places, or—what comes to the same thing—to evaluate the times of events occurring at places remote from the watch.

We might, of course, content ourselves with time values determined by an observer stationed together with the watch at the origin of the co-ordinates, and co-ordinating the corresponding positions of the hands with light signals, given out by every event to be timed, and reaching him through empty space. But this co-ordination has the disadvantage that it is not independent of the standpoint of the observer with the watch or clock, as we know from experience.
We arrive at a much more practical determination along the following line of thought.

If at the point A of space there is a clock, an observer at A can determine the time values of events in the immediate proximity of A by finding the positions of the hands which are simultaneous with these events. If there is at the point B of space another clock in all respects resembling the one at A, it is possible for an observer at B to determine the time values of events in the immediate neighbourhood of B. But it is not possible without further assumption to compare, in respect of time, an event at A with an event at B. We have so far defined only an “A time” and a “B time.” We have not defined a common “time” for A and B, for the latter cannot be defined at all unless we establish by definition that the “time” required by light to travel from A to B equals the “time” it requires to travel from B to A. Let a ray of light start at the “A time” $t_A$ from A towards B, let it at the “B time” $t_B$ be reflected at B in the direction of A, and arrive again at A at the “A time” $t'_A$.

In accordance with definition the two clocks synchronize if

$$t_B - t_A = t'_A - t_B.$$
We assume that this definition of synchronism is free from contradictions, and possible for any number of points; and that the following relations are universally valid:—

1. If the clock at B synchronizes with the clock at A, the clock at A synchronizes with the clock at B.

2. If the clock at A synchronizes with the clock at B and also with the clock at C, the clocks at B and C also synchronize with each other.

Thus with the help of certain imaginary physical experiments we have settled what is to be understood by synchronous stationary clocks located at different places, and have evidently obtained a definition of “simultaneous,” or “synchronous,” and of “time.” The “time” of an event is that which is given simultaneously with the event by a stationary clock located at the place of the event, this clock being synchronous, and indeed synchronous for all time determinations, with a specified stationary clock.

In agreement with experience we further assume the quantity

\[ \frac{2AB}{t'_A - t_A} = c, \]

to be a universal constant—the velocity of light in empty space.

It is essential to have time defined by means of stationary clocks in the stationary system, and the time now defined being appropriate to the stationary system we call it “the time of the stationary system.”
§ 2. On the Relativity of Lengths and Times

The following reflexions are based on the principle of relativity and on the principle of the constancy of the velocity of light. These two principles we define as follows:—

1. The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two systems of co-ordinates in uniform translatory motion.

2. Any ray of light moves in the "stationary" system of co-ordinates with the determined velocity $c$, whether the ray be emitted by a stationary or by a moving body. Hence

$$\text{velocity} = \frac{\text{light path}}{\text{time interval}}$$

where time interval is to be taken in the sense of the definition in § 1.
Let there be given a stationary rigid rod; and let its length be $l$ as measured by a measuring-rod which is also stationary. We now imagine the axis of the rod lying along the axis of $x$ of the stationary system of co-ordinates, and that a uniform motion of parallel translation with velocity $v$ along the axis of $x$ in the direction of increasing $x$ is then imparted to the rod. We now inquire as to the length of the moving rod, and imagine its length to be ascertained by the following two operations:—

(a) The observer moves together with the given measuring-rod and the rod to be measured, and measures the length of the rod directly by superposing the measuring-rod, in just the same way as if all three were at rest.

(b) By means of stationary clocks set up in the stationary system and synchronizing in accordance with § 1, the observer ascertains at what points of the stationary system the two ends of the rod to be measured are located at a definite time. The distance between these two points, measured by the measuring-rod already employed, which in this case is at rest, is also a length which may be designated “the length of the rod.”
In accordance with the principle of relativity the length to be discovered by the operation \((a)\)—we will call it “the length of the rod in the moving system”—must be equal to the length \(l\) of the stationary rod.

The length to be discovered by the operation \((b)\) we will call “the length of the (moving) rod in the stationary system.” This we shall determine on the basis of our two principles, and we shall find that it differs from \(l\).

Current kinematics tacitly assumes that the lengths determined by these two operations are precisely equal, or in other words, that a moving rigid body at the epoch \(t\) may in geometrical respects be perfectly represented by the same body at rest in a definite position.

We imagine further that at the two ends A and B of the rod, clocks are placed which synchronize with the clocks of the stationary system, that is to say that their indications correspond at any instant to the “time of the stationary system” at the places where they happen to be. These clocks are therefore “synchronous in the stationary system.”
We imagine further that with each clock there is a moving observer, and that these observers apply to both clocks the criterion established in § 1 for the synchronization of two clocks. Let a ray of light depart from A at the time \( t_A \), let it be reflected at B at the time \( t_B \), and reach A again at the time \( t'_A \). Taking into consideration the principle of the constancy of the velocity of light we find that

\[
t_B - t_A = \frac{r_{AB}}{c - v} \quad \text{and} \quad t'_A - t_B = \frac{r_{AB}}{c + v}
\]

where \( r_{AB} \) denotes the length of the moving rod—measured in the stationary system. Observers moving with the moving rod would thus find that the two clocks were not synchronous, while observers in the stationary system would declare the clocks to be synchronous.

So we see that we cannot attach any absolute signification to the concept of simultaneity, but that two events which, viewed from a system of co-ordinates, are simultaneous, can no longer be looked upon as simultaneous events when envisaged from a system which is in motion relatively to that system.
II. ELECTRODYNAMICAL PART

§ 6. Transformation of the Maxwell-Hertz Equations for Empty Space. On the Nature of the Electromotive Forces Occurring in a Magnetic Field During Motion

Let the Maxwell-Hertz equations for empty space hold good for the stationary system K, so that we have

\[
\frac{1}{c} \frac{\partial X}{\partial t} = \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z}, \quad \frac{1}{c} \frac{\partial L}{\partial t} = \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y},
\]

\[
\frac{1}{c} \frac{\partial Y}{\partial t} = \frac{\partial L}{\partial z} - \frac{\partial N}{\partial x}, \quad \frac{1}{c} \frac{\partial M}{\partial t} = \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z},
\]

\[
\frac{1}{c} \frac{\partial Z}{\partial t} = \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}, \quad \frac{1}{c} \frac{\partial N}{\partial t} = \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x},
\]

where \((X, Y, Z)\) denotes the vector of the electric force, and \((L, M, N)\) that of the magnetic force.
If we apply to these equations the transformation developed in § 3, by referring the electromagnetic processes to the system of co-ordinates there introduced, moving with the velocity \( v \), we obtain the equations

\[
\begin{align*}
\frac{1}{c} \frac{\partial X}{\partial \tau} &= \frac{\partial}{\partial \eta} \left\{ \beta \left( N - \frac{v}{c} Y \right) \right\} - \frac{\partial}{\partial \zeta} \left\{ \beta \left( M + \frac{v}{c} Z \right) \right\}, \\
\frac{1}{c} \frac{\partial}{\partial \tau} \left\{ \beta \left( Y - \frac{v}{c} N \right) \right\} &= \frac{\partial L}{\partial \zeta}, \\
\frac{1}{c} \frac{\partial}{\partial \tau} \left\{ \beta \left( Z + \frac{v}{c} M \right) \right\} &= \frac{\partial}{\partial \zeta} \left\{ \beta \left( M + \frac{v}{c} Z \right) \right\} - \frac{\partial}{\partial \eta} \left\{ \beta \left( N - \frac{v}{c} Y \right) \right\}, \\
\frac{1}{c} \frac{\partial L}{\partial \tau} &= \frac{\partial}{\partial \zeta} \left\{ \beta \left( Y - \frac{v}{c} N \right) \right\} - \frac{\partial}{\partial \eta} \left\{ \beta \left( Z + \frac{v}{c} M \right) \right\}, \\
\frac{1}{c} \frac{\partial}{\partial \tau} \left\{ \beta \left( M + \frac{v}{c} Z \right) \right\} &= \frac{\partial}{\partial \zeta} \left\{ \beta \left( Z + \frac{v}{c} M \right) \right\} - \frac{\partial X}{\partial \zeta}, \\
\frac{1}{c} \frac{\partial}{\partial \tau} \left\{ \beta \left( N - \frac{v}{c} Y \right) \right\} &= \frac{\partial X}{\partial \eta} - \frac{\partial}{\partial \zeta} \left\{ \beta \left( Y - \frac{v}{c} N \right) \right\},
\end{align*}
\]

where

\[
\beta = \frac{1}{\sqrt{1 - v^2/c^2}}.
\]
Now the principle of relativity requires that if the Maxwell-Hertz equations for empty space hold good in system $K$, they also hold good in system $k$; that is to say that the vectors of the electric and the magnetic force—$(X', Y', Z')$ and $(L', M', N')$—of the moving system $k$, which are defined by their ponderomotive effects on electric or magnetic masses respectively, satisfy the following equations:

\[
\begin{align*}
\frac{1}{c} \frac{\partial X'}{\partial \tau} &= \frac{\partial N'}{\partial \eta} - \frac{\partial M'}{\partial \zeta}, & \frac{1}{c} \frac{\partial L'}{\partial \tau} &= \frac{\partial Y'}{\partial \zeta} - \frac{\partial Z'}{\partial \eta}, \\
\frac{1}{c} \frac{\partial Y'}{\partial \tau} &= \frac{\partial L'}{\partial \zeta} - \frac{\partial N'}{\partial \xi}, & \frac{1}{c} \frac{\partial M'}{\partial \tau} &= \frac{\partial Z'}{\partial \xi} - \frac{\partial X'}{\partial \zeta}, \\
\frac{1}{c} \frac{\partial Z'}{\partial \tau} &= \frac{\partial M'}{\partial \xi} - \frac{\partial L'}{\partial \eta}, & \frac{1}{c} \frac{\partial N'}{\partial \tau} &= \frac{\partial X'}{\partial \eta} - \frac{\partial Y'}{\partial \xi}.
\end{align*}
\]
\[ X' = X, \quad L' = L, \]
\[ Y' = \beta \left( Y - \frac{v}{c}N \right), \quad M' = \beta \left( M + \frac{v}{c}Z \right), \]
\[ Z' = \beta \left( Z + \frac{v}{c}M \right), \quad N' = \beta \left( N - \frac{v}{c}Y \right). \]

As to the interpretation of these equations we make the following remarks: Let a point charge of electricity have the magnitude “one” when measured in the stationary system \( K \), i.e. let it when at rest in the stationary system exert a force of one dyne upon an equal quantity of electricity at a distance of one cm. By the principle of relativity this electric charge is also of the magnitude “one” when measured in the moving system. If this quantity of electricity is at rest relatively to the stationary system, then by definition the vector \( (X, Y, Z) \) is equal to the force acting upon it. If the quantity of electricity is at rest relatively to the moving system (at least at the relevant instant), then the force acting upon it, measured in the moving system, is equal to the vector \( (X', Y', Z') \).
Consequently the first three equations above allow themselves to be clothed in words in the two following ways:—

1. If a unit electric point charge is in motion in an electromagnetic field, there acts upon it, in addition to the electric force, an “electromotive force” which, if we neglect the terms multiplied by the second and higher powers of \( v/c \), is equal to the vector-product of the velocity of the charge and the magnetic force, divided by the velocity of light. (Old manner of expression.)

2. If a unit electric point charge is in motion in an electromagnetic field, the force acting upon it is equal to the electric force which is present at the locality of the charge, and which we ascertain by transformation of the field to a system of co-ordinates at rest relatively to the electrical charge. (New manner of expression.)

The analogy holds with “magnetomotive forces.” We see that electromotive force plays in the developed theory merely the part of an auxiliary concept, which owes its introduction to the circumstance that electric and magnetic forces do not exist independently of the state of motion of the system of co-ordinates.

Furthermore it is clear that the asymmetry mentioned in the introduction as arising when we consider the currents produced by the relative motion of a magnet and a conductor, now disappears. Moreover, questions as to the “seat” of electrodynamic electromotive forces (unipolar machines) now have no point.
Unlike Newtonian mechanics, classical electrodynamics is already consistent with special relativity. Maxwell’s equations and the Lorentz force law can be applied legitimately in any inertial system. Of course, what one observer interprets as an electrical process another may regard as magnetic, but the actual particle motions they predict will be identical. To the extent that this did not work out for Lorentz and others, who studied the question in the late nineteenth century, the fault lay with the nonrelativistic mechanics they used, not with the electrodynamics. Having corrected Newtonian mechanics, we are now in a position to develop a complete and consistent formulation of relativistic electrodynamics. But I emphasize that we will not be changing the rules of electrodynamics in the slightest—rather, we will be expressing these rules in a notation that exposes and illuminates their relativistic character. As we go along, I shall pause now and then to rederive, using the Lorentz transformations, results obtained earlier by more laborious means. But the main purpose of this section is to provide you with a deeper understanding of the structure of electrodynamics—laws that had seemed arbitrary and unrelated before take on a kind of coherence and inevitability when approached from the point of view of relativity.
To begin with I’d like to show you why there had to be such a thing as magnetism, given electrostatics and relativity, and how, in particular, you can calculate the magnetic force between a current-carrying wire and a moving charge without ever invoking the laws of magnetism.¹⁴ Suppose you had a string of positive charges moving along to the right at speed \( u \). I’ll assume the charges are close enough together so that we may regard them as a continuous line charge \( \lambda \). Superimposed on this positive string is a negative one, \(-\lambda\) proceeding to the left at the same speed \( v \). We have, then, a net current to the right, of magnitude

\[
I = 2\lambda v. \tag{12.75}
\]

Meanwhile, a distance \( s \) away there is a point charge \( q \) traveling to the right at speed \( u < v \) (Fig. 12.34a). Because the two line charges cancel, there is no electrical force on \( q \) in this system (\( S \)).
However, let's examine the same situation from the point of view of system $\bar{S}$, which moves to the right with speed $u$ (Fig. 12.34b). In this reference frame $q$ is at rest. By the Einstein velocity addition rule, the velocities of the positive and negative lines are now

$$v_\pm = \frac{v \mp u}{1 \mp vu/c^2}.$$  \hspace{1cm} (12.76)

Because $v_-$ is greater than $v_+$, the Lorentz contraction of the spacing between negative charges is more severe than that between positive charges; *in this frame*, therefore, *the wire carries a net negative charge!* In fact,

$$\lambda_\pm = \pm (\gamma_\pm) \lambda_0,$$  \hspace{1cm} (12.77)

where

$$\gamma_\pm = \frac{1}{\sqrt{1 - v_\pm^2/c^2}}.$$  \hspace{1cm} (12.78)
Figure 12.34
12.3.2 How the Fields Transform

We have learned, in various special cases, that one observer’s electric field is another’s magnetic field. It would be nice to know the general transformation rules for electromagnetic fields: Given the fields in $\mathcal{S}$, what are the fields in $\mathcal{S}'$? Your first guess might be that $E$ is the spatial part of one 4-vector and $B$ the spatial part of another. If so, your intuition is wrong—it’s more complicated than that. Let me begin by making explicit an assumption that was already used implicitly in Sect. 12.3.1: Charge is invariant. Like mass, but unlike energy, the charge of a particle is a fixed number, independent of how fast it happens to be moving. We shall assume also that the transformation rules are the same no matter how the fields were produced—electric fields generated by changing magnetic fields transform the same way as those set up by stationary charges. Were this not the case we’d have to abandon the field formulation altogether, for it is the essence of a field theory that the fields at a given point tell you all there is to know, electromagnetically, about that point; you do not have to append extra information regarding their source.
With this in mind, consider the simplest possible electric field: the uniform field in the region between the plates of a large parallel-plate capacitor (Fig. 12.35a). Say the capacitor is at rest in $S_0$ and carries surface charges $\pm \sigma_0$. Then

$$E_0 = \frac{\sigma_0}{\varepsilon_0} \hat{y}.$$  \hspace{1cm} (12.86)

But what if we examine this same capacitor from system $S$, moving to the right at speed $v_0$ (Fig. 12.35b)? In this system the plates are moving to the left, but the field still takes the form

$$E = \frac{\sigma}{\varepsilon_0} \hat{y};$$  \hspace{1cm} (12.87)

the only difference is the value of the surface charge $\sigma$. 
Now, the total charge on each plate is invariant, and the width \( w \) is unchanged, but the length \( l \) is Lorentz-contrasted by a factor

\[
\frac{1}{\gamma_0} = \sqrt{1 - \frac{v^2}{c^2}},
\]

so the charge per unit area is increased by a factor \( \gamma_0 \):

\[
\sigma = \gamma_0 \sigma_0.
\]

Accordingly,

\[
E^\perp = \gamma_0 E_0^\perp.
\]

I have put in the superscript \( \perp \) to make it clear that this rule pertains to components of \( \mathbf{E} \) that are perpendicular to the direction of motion of \( S \). To get the rule for parallel components, consider the capacitor lined up with the \( yz \) plane (Fig. 12.36). This time it is the plate separation \( d \) that is Lorentz-contrasted, whereas \( l \) and \( w \) (and hence also \( \sigma \)) are the same in both frames. Since the field does not depend on \( d \), it follows that

\[
E^\parallel = E_0^\parallel.
\]
But Eqs. 12.90 and 12.91 are not the most general transformation laws, for we began with a system \( S_0 \) in which the charges were at rest and where, consequently, there was no magnetic field. To derive the \textit{general} rule we must start out in a system with both electric and magnetic fields. For this purpose \( S \) itself will serve nicely. In addition to the electric field

\[ E_y = \frac{\sigma}{\epsilon_0}, \]  

(12.93)

there is a \textit{magnetic} field due to the surface currents (Fig. 12.35b):

\[ \mathbf{K}_\pm = \mp \sigma v_0 \hat{x}. \]

(12.94)

By the right-hand rule, this field points in the negative \( z \) direction; its magnitude is given by Ampère’s law:

\[ B_z = -\mu_0 \sigma v_0. \]

(12.95)
In a third system, $\tilde{S}$, traveling to the right with speed $v$ relative to $S$ (Fig. 12.38), the fields would be

$$\tilde{E}_y = \frac{\bar{\sigma}}{\varepsilon_0}, \quad \tilde{B}_z = -\mu_0 \bar{\sigma} \bar{v},$$

(12.96)

where $\bar{v}$ is the velocity of $\tilde{S}$ relative to $S_0$:

$$\bar{v} = \frac{v + v_0}{1 + vv_0/c^2}, \quad \bar{\gamma} = \frac{1}{\sqrt{1 - \bar{v}^2/c^2}},$$

(12.97)

and

$$\bar{\sigma} = \bar{\gamma} \sigma_0.$$  

(12.98)
It remains only to express $\vec{E}$ and $\vec{B}$ (Eq. 12.96), in terms of $E$ and $B$ (Eqs. 12.93 and 12.95). In view of Eqs. 12.89 and 12.98, we have

$$\tilde{E}_y = \left( \frac{\tilde{\gamma}}{\gamma_0} \right) \frac{\sigma}{\varepsilon_0}, \quad \tilde{B}_z = -\left( \frac{\tilde{\gamma}}{\gamma_0} \right) \mu_0 \sigma \tilde{v}. \quad (12.99)$$

With a little algebra, you will find that

$$\frac{\tilde{\gamma}}{\gamma_0} = \sqrt{\frac{1 - v^2/c^2}{1 - \tilde{v}^2/c^2}} = \frac{1 + vv_0/c^2}{\sqrt{1 - v^2/c^2}} = \gamma \left( 1 + \frac{vv_0}{c^2} \right), \quad (12.100)$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \quad (12.101)$$

as always. Thus,

$$\tilde{E}_y = \gamma \left( 1 + \frac{vv_0}{c^2} \right) \frac{\sigma}{\varepsilon_0} = \gamma \left( E_y - \frac{v}{c^2 \varepsilon_0 \mu_0} B_z \right),$$

whereas

$$\tilde{B}_z = -\gamma \left( 1 + \frac{vv_0}{c^2} \right) \mu_0 \sigma \left( \frac{v + v_0}{1 + vv_0/c^2} \right) = \gamma (B_z - \mu_0 \varepsilon_0 v E_y).$$

Or, since $\mu_0 \varepsilon_0 = 1/c^2$,

$$\begin{align*}
\tilde{E}_y &= \gamma (E_y - v B_z), \\
\tilde{B}_z &= \gamma \left( B_z - \frac{v}{c^2} E_y \right). \quad (12.102)
\end{align*}$$
This tells us how \( E_y \) and \( B_z \) transform—to do \( E_z \) and \( B_y \) we simply align the same capacitor parallel to the \( xy \) plane instead of the \( xz \) plane (Fig. 12.39). The fields in \( S \) are then

\[
E_z = \frac{\sigma}{\epsilon_0}, \quad B_y = \mu_0 \sigma v_0.
\]

(Use the right-hand rule to get the sign of \( B_y \).) The rest of the argument is identical—everywhere we had \( E_y \) before, read \( E_z \), and everywhere we had \( B_z \), read \(-B_y\):

\[
\begin{align*}
\tilde{E}_z &= \gamma (E_z + v B_y), \\
\tilde{B}_y &= \gamma \left( B_y + \frac{v}{c^2} E_z \right).
\end{align*}
\]  

(12.103)

As for the \( x \) components, we have already seen (by orienting the capacitor parallel to the \( yz \) plane) that

\[
\tilde{E}_x = E_x.
\]  

(12.104)
Since in this case there is no accompanying magnetic field, we cannot deduce the transformation rule for $B_x$. But another configuration will do the job: Imagine a long *solenoid* aligned parallel to the $x$ axis (Fig. 12.40) and at rest in $S$. The magnetic field within the coil is

$$B_x = \mu_0 n I,$$  \hspace{1cm} (12.105)

where $n$ is the number of turns per unit length, and $I$ is the current. In system $\tilde{S}$, the length contracts, so $n$ increases:

$$\tilde{n} = \gamma n.$$  \hspace{1cm} (12.106)
On the other hand, time \textit{dilates}: The $S$ clock, which rides along with the solenoid, runs slow, so the current (charge \textit{per unit time}) in $\tilde{S}$ is given by

\[ \tilde{I} = \frac{1}{\gamma} I. \]  

(12.107)

The two factors of $\gamma$ exactly cancel, and we conclude that

\[ \tilde{B}_x = B_x. \]

Like $E$, the component of $B$ \textit{parallel} to the motion is unchanged.

Let’s now collect together the complete set of transformation rules:

\[
\begin{align*}
\tilde{E}_x &= E_x, & \tilde{E}_y &= \gamma (E_y - vB_z), & \tilde{E}_z &= \gamma (E_z + vB_y), \\
\tilde{B}_x &= B_x, & \tilde{B}_y &= \gamma \left( B_y + \frac{v}{c^2} E_z \right), & \tilde{B}_z &= \gamma \left( B_z - \frac{v}{c^2} E_y \right).
\end{align*}
\]  

(12.108)
Two special cases warrant particular attention:

1. If \( \mathbf{B} = 0 \) in \( S \), then

\[
\mathbf{\bar{B}} = \gamma \frac{v}{c^2} (E_z \hat{y} - E_y \hat{z}) = \frac{v}{c^2} (\mathbf{\bar{E}}_z \hat{y} - \mathbf{\bar{E}}_y \hat{z}),
\]

or, since \( v = v \hat{x} \),

\[
\mathbf{\bar{B}} = -\frac{1}{c^2} (v \times \mathbf{\bar{E}}).
\]  \hspace{1cm} (12.109)

2. If \( \mathbf{E} = 0 \) in \( S \), then

\[
\mathbf{\bar{E}} = -\gamma v(B_z \hat{y} - B_y \hat{z}) = -v(B_z \hat{y} - B_y \hat{z}),
\]

or

\[
\mathbf{\bar{E}} = v \times \mathbf{\bar{B}}.
\]  \hspace{1cm} (12.110)

In other words, if either \( \mathbf{E} \) or \( \mathbf{B} \) is zero (at a particular point) in one system, then in any other system the fields (at that point) are very simply related by Eq. 12.109 or Eq. 12.110.
§ 7. Theory of Doppler’s Principle and of Aberration

In the system $K$, very far from the origin of co-ordinates, let there be a source of electrodynamic waves, which in a part of space containing the origin of co-ordinates may be represented to a sufficient degree of approximation by the equations

\[
X = X_0 \sin \Phi, \quad L = L_0 \sin \Phi, \\
Y = Y_0 \sin \Phi, \quad M = M_0 \sin \Phi, \\
Z = Z_0 \sin \Phi, \quad N = N_0 \sin \Phi,
\]

where

\[
\Phi = \omega \left\{ t - \frac{1}{c} (lx + my + nz) \right\}.
\]

Here $(X_0, Y_0, Z_0)$ and $(L_0, M_0, N_0)$ are the vectors defining the amplitude of the wave-train, and $l, m, n$ the direction-cosines of the wave-normals. We wish to know the constitution of these waves, when they are examined by an observer at rest in the moving system $k$. 
Applying the equations of transformation found in § 6 for electric and magnetic forces, and those found in § 3 for the co-ordinates and the time, we obtain directly

\[ X' = X_0 \sin \Phi', \quad L' = L_0 \sin \Phi', \]
\[ Y' = \beta(Y_0 - vN_0/c) \sin \Phi', \quad M' = \beta(M_0 + vZ_0/c) \sin \Phi', \]
\[ Z' = \beta(Z_0 + vM_0/c) \sin \Phi', \quad N' = \beta(N_0 - vY_0/c) \sin \Phi', \]
\[ \Phi' = \omega' \left\{ \tau - \frac{1}{c} (l' \xi + m' \eta + n' \zeta) \right\} \]

where
\[ \omega' = \frac{\omega \beta(1 - lv/c)}{1 - lv/c}, \]
\[ l' = \frac{l - v/c}{1 - lv/c}, \]
\[ m' = \frac{m}{\beta(1 - lv/c)}, \]
\[ n' = \frac{n}{\beta(1 - lv/c)}. \]
From the equation for $\omega'$ it follows that if an observer is moving with velocity $v$ relatively to an infinitely distant source of light of frequency $\nu$, in such a way that the connecting line “source-observer” makes the angle $\phi$ with the velocity of the observer referred to a system of co-ordinates which is at rest relatively to the source of light, the frequency $\nu'$ of the light perceived by the observer is given by the equation

$$\nu' = \nu \frac{1 - \cos \phi \cdot v/c}{\sqrt{1 - v^2/c^2}}.$$

This is Doppler’s principle for any velocities whatever. When $\phi = 0$ the equation assumes the perspicuous form

$$\nu' = \nu \sqrt{\frac{1 - v/c}{1 + v/c}}.$$

We see that, in contrast with the customary view, when $v = -c$, $\nu' = \infty$. 
We still have to find the amplitude of the waves, as it appears in the moving system. If we call the amplitude of the electric or magnetic force $A$ or $A'$ respectively, accordingly as it is measured in the stationary system or in the moving system, we obtain

$$A'^2 = A^2 \frac{(1 - \cos \phi \cdot v/c)^2}{1 - v^2/c^2}$$

which equation, if $\phi = 0$, simplifies into

$$A'^2 = A^2 \frac{1 - v/c}{1 + v/c}.$$

It follows from these results that to an observer approaching a source of light with the velocity $c$, this source of light must appear of infinite intensity.
§ 8. Transformation of the Energy of Light Rays. Theory of the Pressure of Radiation Exerted on Perfect Reflectors

If $S$ is the volume of the sphere, and $S'$ that of this ellipsoid, then by a simple calculation

$$\frac{S'}{S} = \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 - \cos \phi \cdot \frac{v}{c}}.$$

Thus, if we call the light energy enclosed by this surface $E$ when it is measured in the stationary system, and $E'$ when measured in the moving system, we obtain

$$\frac{E'}{E} = \frac{A'^2 S'}{A^2 S} = \frac{1 - \cos \phi \cdot \frac{v}{c}}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

and this formula, when $\phi = 0$, simplifies into

$$\frac{E'}{E} = \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}.$$

It is remarkable that the energy and the frequency of a light complex vary with the state of motion of the observer in accordance with the same law.
DOES THE INERTIA OF A BODY DEPEND UPON ITS ENERGY-CONTENT?

By A. EINSTEIN

September 27, 1905

The results of the previous investigation lead to a very interesting conclusion, which is here to be deduced.

I based that investigation on the Maxwell-Hertz equations for empty space, together with the Maxwellian expression for the electromagnetic energy of space, and in addition the principle that:

The laws by which the states of physical systems alter are independent of the alternative, to which of two systems of coordinates, in uniform motion of parallel translation relatively to each other, these alterations of state are referred (principle of relativity).

With these principles* as my basis I deduced inter alia the following result (§ 8):

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*The principle of the constancy of the velocity of light is of course contained in Maxwell’s equations.
With these principles* as my basis I deduced \textit{inter alia} the following result (§ 8):

Let a system of plane waves of light, referred to the system of co-ordinates \((x, y, z)\), possess the energy \(l\); let the direction of the ray (the wave-normal) make an angle \(\phi\) with the axis of \(x\) of the system. If we introduce a new system of co-ordinates \((\xi, \eta, \zeta)\) moving in uniform parallel translation with respect to the system \((x, y, z)\), and having its origin of co-ordinates in motion along the axis of \(x\) with the velocity \(v\), then this quantity of light—measured in the system \((\xi, \eta, \zeta)\)—possesses the energy

\[
l^* = l \frac{1 - \frac{v}{c} \cos \phi}{\sqrt{1 - v^2/c^2}}
\]

where \(c\) denotes the velocity of light. We shall make use of this result in what follows.
Let there be a stationary body in the system \((x, y, z)\), and let its energy—referred to the system \((x, y, z)\) be \(E_0\). Let the energy of the body relative to the system \((\xi, \eta, \zeta)\) moving as above with the velocity \(v\), be \(H_0\).

Let this body send out, in a direction making an angle \(\phi\) with the axis of \(x\), plane waves of light, of energy \(\frac{1}{2}L\) measured relatively to \((x, y, z)\), and simultaneously an equal quantity of light in the opposite direction. Meanwhile the body remains at rest with respect to the system \((x, y, z)\). The principle of energy must apply to this process, and in fact (by the principle of relativity) with respect to both systems of co-ordinates. If we call the energy of the body after the emission of light \(E_1\) or \(H_1\) respectively, measured relatively to the system \((x, y, z)\) or \((\xi, \eta, \zeta)\) respectively, then by employing the relation given above we obtain

\[
E_0 = E_1 + \frac{1}{2}L + \frac{1}{2}L, \\
H_0 = H_1 + \frac{1}{2}L \frac{1 - \frac{v}{c} \cos \phi}{\sqrt{1 - v^2/c^2}} + \frac{1}{2}L \frac{1 + \frac{v}{c} \cos \phi}{\sqrt{1 - v^2/c^2}} \\
= H_1 + \frac{L}{\sqrt{1 - v^2/c^2}}.
\]

By subtraction we obtain from these equations

\[
H_0 - E_0 - (H_1 - E_1) = L \left\{ \frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right\}.
\]
The two differences of the form $H - E$ occurring in this expression have simple physical significations. $H$ and $E$ are energy values of the same body referred to two systems of co-ordinates which are in motion relatively to each other, the body being at rest in one of the two systems (system $(x, y, z)$). Thus it is clear that the difference $H - E$ can differ from the kinetic energy $K$ of the body, with respect to the other system $(\xi, \eta, \zeta)$, only by an additive constant $C$, which depends on the choice of the arbitrary additive constants of the energies $H$ and $E$. Thus we may place

$$
\begin{align*}
H_0 - E_0 &= K_0 + C, \\
H_1 - E_1 &= K_1 + C,
\end{align*}
$$

since $C$ does not change during the emission of light. So we have

$$K_0 - K_1 = L \left\{ \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right\}.$$

The kinetic energy of the body with respect to $(\xi, \eta, \zeta)$ diminishes as a result of the emission of light, and the amount of diminution is independent of the properties of the body. Moreover, the difference $K_0 - K_1$, like the kinetic energy of the electron (§ 10), depends on the velocity.
Neglecting magnitudes of fourth and higher orders we may place

\[ K_0 - K_1 = \frac{1}{2} \frac{L}{c^2} v^2. \]

From this equation it directly follows that:

*If a body gives off the energy \( L \) in the form of radiation, its mass diminishes by \( L/c^2 \).* The fact that the energy withdrawn from the body becomes energy of radiation evidently makes no difference, so that we are led to the more general conclusion that

The mass of a body is a measure of its energy-content; if the energy changes by \( L \), the mass changes in the same sense by \( L/9 \times 10^{20} \), the energy being measured in ergs, and the mass in grammes.

It is not impossible that with bodies whose energy-content is variable to a high degree (e.g. with radium salts) the theory may be successfully put to the test.

If the theory corresponds to the facts, radiation conveys inertia between the emitting and absorbing bodies.
12.1.4 The Structure of Spacetime

(i) Four-vectors. The Lorentz transformations take on a simpler appearance when expressed in terms of the quantities

\[ x^0 \equiv ct, \quad \beta \equiv \frac{v}{c}. \]  

(12.21)

Using \( x^0 \) (instead of \( t \)) and \( \beta \) (instead of \( v \)) amounts to changing the unit of time from the second to the meter—1 meter of \( x^0 \) corresponds to the time it takes light to travel 1 meter (in vacuum). If, at the same time, we number the \( x, y, z \) coordinates, so that

\[ x^1 = x, \quad x^2 = y, \quad x^3 = z, \]  

(12.22)

then the Lorentz transformations read

\[
\begin{align*}
\bar{x}^0 &= \gamma (x^0 - \beta x^1), \\
\bar{x}^1 &= \gamma (x^1 - \beta x^0), \\
\bar{x}^2 &= x^2, \\
\bar{x}^3 &= x^3.
\end{align*}
\]  

(12.23)
Or, in matrix form:

\[
\begin{pmatrix}
\bar{x}^0 \\
\bar{x}^1 \\
\bar{x}^2 \\
\bar{x}^3 \\
\end{pmatrix} =
\begin{pmatrix}
\gamma & -\gamma \beta & 0 & 0 \\
-\gamma \beta & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
x^0 \\
x^1 \\
x^2 \\
x^3 \\
\end{pmatrix}.
\]

(12.24)

Letting Greek indices run from 0 to 3, this can be distilled into a single equation:

\[
\bar{x}^\mu = \sum_{\nu=0}^{3} (\Lambda^\mu_\nu) x^\nu,
\]

(12.25)

where $\Lambda$ is the **Lorentz transformation matrix** in Eq. 12.24 (the superscript $\mu$ labels the row, the subscript $\nu$ labels the column). One virtue of writing things in this abstract manner is that we can handle in the same format a more general transformation, in which the relative motion is not along a common $x\bar{x}$ axis; the matrix $\Lambda$ would be more complicated, but the structure of Eq. 12.25 is unchanged.
under rotations the same way \((x, y, z)\) do; by extension, we now define a 4-vector as any set of four components that transform in the same manner as \((x^0, x^1, x^2, x^3)\) under Lorentz transformations:

\[
\bar{a}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu_{\nu} a^\nu.
\]  

(12.26)

For the particular case of a transformation along the \(x\) axis:

\[
\begin{align*}
\bar{a}^0 &= \gamma (a^0 - \beta a^1), \\
\bar{a}^1 &= \gamma (a^1 - \beta a^0), \\
\bar{a}^2 &= a^2, \\
\bar{a}^3 &= a^3.
\end{align*}
\]

(12.27)
There is a 4-vector analog to the dot product \((\mathbf{A} \cdot \mathbf{B} \equiv A_x B_x + A_y B_y + A_z B_z)\), but it’s not just the sum of the products of like components; rather, the zeroth components have a minus sign:

\[
-a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3. \tag{12.28}
\]

This is the **four-dimensional scalar product**; you should check for yourself (Prob. 12.17) that it has the same value in all inertial systems:

\[
-a^0 \bar{b}^0 + \bar{a}^1 b^1 + \bar{a}^2 b^2 + \bar{a}^3 b^3 = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3. \tag{12.29}
\]

Just as the ordinary dot product is **invariant** (unchanged) under rotations, this combination is invariant under Lorentz transformations.

To keep track of the minus sign it is convenient to introduce the **covariant** vector \(a_\mu\), which differs from the **contravariant** \(a^\mu\) only in the sign of the zeroth component:

\[
a_\mu = (a_0, a_1, a_2, a_3) \equiv (-a^0, a^1, a^2, a^3). \tag{12.30}
\]
You must be scrupulously careful about the placement of indices in this business: *upper* indices designate *contravariant* vectors; *lower* indices are for *covariant* vectors. Raising or lowering the temporal index costs a minus sign \((a_0 = -a^0)\); raising or lowering a spatial index changes nothing \((a_1 = a^1, a_2 = a^2, a_3 = a^3)\). The scalar product can now be written with the summation symbol,

\[
\sum_{\mu=0}^{3} a_\mu b^\mu, \tag{12.31}
\]

or, more compactly still,

\[
a_\mu b^\mu. \tag{12.32}
\]

Summation is *implied* whenever a Greek index is repeated in a product—once as a covariant index and once as contravariant. This is called the **Einstein summation convention**, after its inventor, who regarded it as one of his most important contributions. Of course, we could as well take care of the minus sign by switching to covariant \(b\):

\[
a_\mu b^\mu = a^\mu b_\mu = -a_0^0 b^0 + a_1^1 b^1 + a_2^2 b^2 + a_3^3 b^3. \tag{12.33}
\]
(ii) The invariant interval. Suppose event $A$ occurs at $(x_A^0, x_A^1, x_A^2, x_A^3)$, and event $B$ at $(x_B^0, x_B^1, x_B^2, x_B^3)$. The difference,

$$\Delta x^\mu \equiv x_A^\mu - x_B^\mu,$$

(12.35)

is the displacement 4-vector. The scalar product of $\Delta x^\mu$ with itself is a quantity of special importance; we call it the interval between two events:

$$I \equiv (\Delta x)_\mu (\Delta x)^\mu = - (\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 = -c^2 t^2 + d^2,$$

(12.36)

where $t$ is the time difference between the two events and $d$ is their spatial separation. When you transform to a moving system, the time between $A$ and $B$ is altered ($\tilde{t} \neq t$), and so is the spatial separation ($\tilde{d} \neq d$), but the interval $I$ remains the same.
Depending on the two events in question, the interval can be positive, negative, or zero:

1. If \( I < 0 \) we call the interval **timelike**, for this is the sign we get when the two occur at the same place \((d = 0)\), and are separated only temporally.

2. If \( I > 0 \) we call the interval **spacelike**, for this is the sign we get when the two occur at the same time \((t = 0)\), and are separated only spatially.

3. If \( I = 0 \) we call the interval **lightlike**, for this is the relation that holds when the two events are connected by a signal traveling at the speed of light.

If the interval between the two events is timelike, there exists an inertial system (accessible by Lorentz transformation) in which they occur at the same point. For if I hop on a train going from \((A)\) to \((B)\) at the speed \(v = d/t\), leaving event \(A\) when it occurs, I shall be just in time to pass \(B\) when it occurs; in the train system, \(A\) and \(B\) take place at the same point. You cannot do this for a **spacelike** interval, of course, because \(v\) would have to be greater than \(c\), and no observer can exceed the speed of light (\(\gamma\) would be imaginary and the Lorentz transformations would be nonsense). On the other hand, if the interval is spacelike, then there exists a system in which the two events occur at the same time (see Prob. 12.21).
(iii) **Space-time diagrams.** If you want to represent the motion of a particle graphically, the normal practice is to plot the position versus time (that is, \( x \) runs vertically and \( t \) horizontally). On such a graph, the velocity can be read off as the slope of the curve. For some reason the convention is reversed in relativity: everyone plots position horizontally and time (or, better, \( x^0 = ct \)) vertically. Velocity is then given by the *reciprocal* of the slope. A particle at rest is represented by a vertical line; a photon, traveling at the speed of light, is described by a 45° line; and a rocket going at some intermediate speed follows a line of slope \( c/v = 1/\beta \) (Fig. 12.21). We call such plots **Minkowski diagrams.**

The trajectory of a particle on a Minkowski diagram is called a **world line.** Suppose you set out from the origin at time \( t = 0 \). Because no material object can travel faster than light, your world line can never have a slope less than 1. Accordingly, your motion is restricted to the wedge-shaped region bounded by the two 45° lines (Fig. 12.22). We call
restricted to the wedge-shaped region bounded by the two 45° lines (Fig. 12.22). We call this your “future,” in the sense that it is the locus of all points accessible to you. Of course, as time goes on, and you move along your chosen world line, your options progressively narrow: your “future” at any moment is the forward “wedge” constructed at whatever point you find yourself. Meanwhile, the backward wedge represents your “past,” in the sense that it is the locus of all points from which you might have come. As for the rest (the region outside the forward and backward wedges) this is the generalized “present.” You can’t get there, and you didn’t come from there. In fact, there’s no way can can influence any event in the present (the message would have to travel faster than light); it’s a vast expanse of spacetime that is absolutely inaccessible to you.
12.3.3 The Field Tensor

As Eq. 12.108 indicates, \( \mathbf{E} \) and \( \mathbf{B} \) certainly do not transform like the spatial parts of the two 4-vectors—in fact, the components of \( \mathbf{E} \) and \( \mathbf{B} \) are stirred together when you go from one inertial system to another. What sort of an object is this, which has six components and transforms according to Eq. 12.108? Answer: It’s an **antisymmetric, second-rank tensor**.

Remember that a 4-vector transforms by the rule

\[
\tilde{a}^\mu = \Lambda_\nu^\mu a^\nu
\]  

(12.112)

(summation over \( \nu \) implied), where \( \Lambda \) is the Lorentz transformation matrix. If \( \tilde{S} \) is moving in the \( x \) direction at speed \( v \), \( \Lambda \) has the form

\[
\Lambda = \begin{pmatrix}
\gamma & -\gamma \beta & 0 & 0 \\
-\gamma \beta & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]  

(12.113)

and \( \Lambda_\nu^\mu \) is the entry in row \( \mu \), column \( \nu \). A (second-rank) tensor is an object with two indices, which transform with two factors of \( \Lambda \) (one for each index):

\[
\tilde{t}^{\mu \nu} = \Lambda_\lambda^\mu \Lambda_\sigma^\nu t^{\lambda \sigma}.
\]  

(12.114)
A tensor (in 4 dimensions) has $4 \times 4 = 16$ components, which we can display in a $4 \times 4$ array:

$$t^{\mu\nu} = \begin{pmatrix} t^{00} & t^{01} & t^{02} & t^{03} \\ t^{10} & t^{11} & t^{12} & t^{13} \\ t^{20} & t^{21} & t^{22} & t^{23} \\ t^{30} & t^{31} & t^{32} & t^{33} \end{pmatrix}.$$  

However, the 16 elements need not all be different. For instance, a symmetric tensor has the property

$$t^{\mu\nu} = t^{\nu\mu} \quad \text{(symmetric tensor).} \quad (12.115)$$

In this case there are 10 distinct components; 6 of the 16 are repeats ($t^{01} = t^{10}$, $t^{02} = t^{20}$, $t^{03} = t^{30}$, $t^{12} = t^{21}$, $t^{13} = t^{31}$, $t^{23} = t^{32}$). Similarly, an antisymmetric tensor obeys

$$t^{\mu\nu} = -t^{\nu\mu} \quad \text{(antisymmetric tensor).} \quad (12.116)$$
Such an object has just 6 distinct elements—of the original 16, six are repeats (the same ones as before, only this time with a minus sign) and four are zero \( (t^{00}, t^{11}, t^{22}, \text{ and } t^{33}) \). Thus, the general antisymmetric tensor has the form

\[
t^{\mu\nu} = \begin{pmatrix}
0 & t^{01} & t^{02} & t^{03} \\
-t^{01} & 0 & t^{12} & t^{13} \\
-t^{02} & -t^{12} & 0 & t^{23} \\
-t^{03} & -t^{13} & -t^{23} & 0
\end{pmatrix}.
\]

Let's see how the transformation rule 12.114 works, for the six distinct components of an antisymmetric tensor. Starting with \( \bar{t}^{01} \), we have

\[
\bar{t}^{01} = \Lambda_0^0 \Lambda_1^1 t^{\lambda\sigma},
\]

but according to Eq. 12.113, \( \Lambda_0^0 = 0 \) unless \( \lambda = 0 \) or 1, and \( \Lambda_1^1 = 0 \) unless \( \sigma = 0 \) or 1. So there are four terms in the sum:

\[
\bar{t}^{01} = \Lambda_0^0 \Lambda_0^0 t^{00} + \Lambda_0^0 \Lambda_1^1 t^{01} + \Lambda_1^0 \Lambda_0^1 t^{10} + \Lambda_1^0 \Lambda_1^1 t^{11}.
\]

On the other hand, \( t^{00} = t^{11} = 0 \), while \( t^{01} = -t^{10} \), so

\[
\bar{t}^{01} = (\Lambda_0^0 \Lambda_1^1 - \Lambda_1^0 \Lambda_0^1) t^{01} = (\gamma^2 - (\gamma \beta)^2) t^{01} = t^{01}.
\]
These are precisely the rules we derived on physical grounds for the electromagnetic fields (Eq. 12.108)—in fact, we can construct the field tensor $F^{\mu\nu}$ by direct comparison:\[15\]

\[
F^{01} \equiv \frac{E_x}{c}, \quad F^{02} \equiv \frac{E_y}{c}, \quad F^{03} \equiv \frac{E_z}{c}, \quad F^{12} \equiv B_z, \quad F^{31} \equiv B_y, \quad F^{23} \equiv B_x.
\]

Written as an array,

\[
F^{\mu\nu} = \begin{pmatrix}
0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\
-\frac{E_x}{c} & 0 & B_z & -B_y \\
-\frac{E_y}{c} & -B_z & 0 & B_x \\
-\frac{E_z}{c} & B_y & -B_x & 0
\end{pmatrix}.
\] (12.118)

Thus relativity completes and perfects the job begun by Oersted, combining the electric and magnetic fields into a single entity, $F^{\mu\nu}$.

---

\[15\] Some authors prefer the convention $F^{01} \equiv E_x$, $F^{12} \equiv cB_z$, and so on, and some use the opposite signs. Accordingly, most of the equations from here on will look a little different, depending on the text.
If you followed that argument with exquisite care, you may have noticed that there was a different way of imbedding $\mathbf{E}$ and $\mathbf{B}$ in an antisymmetric tensor: instead of comparing the first line of Eq. 12.108 with the first line of Eq. 12.117, and the second with the second, we could relate the first line of Eq. 12.108 to the second line of Eq. 12.117, and vice versa. This leads to dual tensor, $G^{\mu\nu}$:

\[
G^{\mu\nu} = \begin{pmatrix}
0 & B_x & B_y & B_z \\
-B_x & 0 & -E_z/c & E_y/c \\
-B_y & E_z/c & 0 & -E_x/c \\
-B_z & -E_y/c & E_x/c & 0
\end{pmatrix}.
\]  

(12.119)

$G^{\mu\nu}$ can be obtained directly from $F^{\mu\nu}$ by the substitution $\mathbf{E}/c \rightarrow \mathbf{B}$, $\mathbf{B} \rightarrow -\mathbf{E}/c$. Notice that this operation leaves Eq. 12.108 unchanged—that’s why both tensors generate the correct transformation rules for $\mathbf{E}$ and $\mathbf{B}$.